Analytic AGM Revision

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Abstract

Since they were introduced, AGM *revision* and Katsuno and Mendelzon's *update* have been considered esentially different theory change operations serving different purposes. This work provides a new presentation of AGM revision based on the update semantic apparatus establishing in such a way a bridge between the two seemingly incomparable frameworks.

We define a new operation $\bar{\bullet}$ as a variant of the standard update that we call an *analytic revision*. We prove the correspondence between analytic revisions and (transitively relational) AGM revisions when a given fixed theory is considered (Theorem 4.8). Furthermore, we can characterize analytic revision functions for possibly infinite languages as those AGM revisions satisfying (K*1)-(K*8) plus two new postulates (LU-$\exists$) and (LU-$\forall$) governing the revision of different theories (Theorem 5.3).

We believe these results bring new light to the issue of how revision and update functions are related. They also provide a novel way to achieve iterated theory revision.

1 Overview

Since they were introduced, the AGM notion of *revision* [1] and Katsuno and Mendelzon's *update* [11] have been considered two fundamentally different ways of changing knowledge. The usual interpretation is as follows. Revision takes place when new information is regarded as correcting our theory of a *static object*. According to revision, as little as possible of the original theory should be changed in order to accommodate the new information. It is in this sense that revision involves minimal change. Update, in contrast, takes place when our representation is modified in response to changes into an *evolving object*. When new information is confronted, each complete consistent extension of our theory should be considered. From the perspective of each of these extensions, the most similar complete theory accounting for the new information should be chosen. The result of the update is what the new theories have in common. It is in this sense that update involves maximal similarity.

Because of these differences, many researchers have considered revision and update as essentially different operations serving different purposes, and hence incomparable. In this paper we establish a bridge between the two kinds of change. We will provide a new presentation of AGM revision based on the update semantic apparatus. Our strategy will be to semantically define a new operation as a variant of the update operation, that we will dub *analytic revision*. The key idea behind it will be the definition of a meaningful relation obtained from the pointwise relations of the update operation.

In Section 2 we state basic logical conventions and fix notation. We briefly present AGM revision functions and we make a decisive observation on the set of all AGM revision functions. Also in this section we present the update functions, generalizing them for infinite languages. Surprisingly Katsuno and Mendelzon's original postulates (in [11]) turned out to be incomplete for infinite languages. In the next two sections we formally introduce our analytic revision function and investigate its connections with revision and update.

Section 5 is the main of this paper and Theorem 5.3 is the most important result, a representation theorem for analytic revision functions. It provides a characterization of analytic functions as those AGM functions satisfying (K*1)-(K*8) plus two new postulates, ($K{*}\exists$) and ($K{*}\forall$), governing the revision of different theories, for possibly infinite languages.

Finally in Section 6 we discuss the capabilities of the analytic revision function to achieve iterated change, and a number of well known properties of iterated change are examined.

This study builds on one of the author's initial work in [6] and [7]. In the first our current analytic function was called a "lazy update" reflecting that it was semantically defined as a variant of the standard update operation. Lazy updates were just defined for finite languages and we proved they satisfy all AGM revision postulates.

The independent work of Schlechta, Lehmann and Magidor's "Distance Semantics for Belief Revision" [13] turned out to be related to ours. Their revision function based on distances and our analytic revision function are definitionally equivalent, modulo some considerations over the formal structures they are based on. Our work extends and continues theirs in several respects. We consider an infinite language while they don't and we obtain characterization results for functions built over non symmetric distances —

an explicit question left open in [13]. Also novel in our work is the definition of AGM revision in the update semantic structure, which allows us to connect these two seemingly incomparable forms of theory change.

2 Background

2.1 Basic notions and notational conventions

We will assume familiarity with the AGM theory and Katsuno and Mendelzon framework, as well as basic notions in logic. We consider a classical propositional language L and denote with P the set of all its propositional letters. If P is finite we will call L a finite propositional language. The symbols $\wedge, \vee, \neg, \supset, \equiv$ will denote the usual truth functional connectives. Capital letters A, B, C will be used to denote arbitrary formulae of L. We consider Cn a Tarskian consequence operation, and following [1] we assume Cn on L satisfies supra classicality, (if A can be derived from X by classical truth functional logic, then $A \in \mathrm{Cn}(X)$), compactness (if $A \in \mathrm{Cn}(X)$, then $A \in \mathrm{Cn}(Y)$ for some finite subset $Y \subseteq X$) and the rule of introduction of disjunction into the premisses (if $C \in \mathrm{Cn}(X \cup \{A\})$ and $C \in \mathrm{Cn}(X \cup \{B\})$ then $C \in \mathrm{Cn}(X \cup \{A \vee B\})$. Under these assumptions the consequence operation Cn also satisfies the deduction theorem ($B \in \mathrm{Cn}(X \cup \{A\})$ if and only if $(A \supset B) \in \mathrm{Cn}(X)$).

A theory is a subset of L closed under Cn. Capital letters K, K', H are used for theories of L, and we denote by $\mathbb{K}$ the set of all theories of L. While L is the largest theory, $\mathrm{Cn}(\emptyset)$ is the smallest. A subset X of L is consistent (modulo Cn) iff for no formula A do we have $(A \wedge \neg A) \in \mathrm{Cn}(X)$. A theory is complete if it sanctions a truth value for each propositional letter.

We take W as the set of all maximal consistent subsets of L, that is, the set of all complete consistent extensions of L. The valuation function $[\ ] : L \to \mathcal{P}(W)$ is defined as usual, for any propositional letter p, $w \in [p]$ iff $p \in w$. Given $A \in L$ we denote by $[A]$ the proposition for A, or the set of A-worlds, the set of elements of W satisfying A. For the purposes of this work we consider the terms maximal consistent subset of L, valuation on L and possible world, interchangeable. This, of course, amounts to working with models that are injective with respect to the interpretation function (no two distinct worlds satisfy exactly the same formulae) and full (every consistent set of formulae is satisfied by some world). If K is a theory, $[K]$ denotes the set of possible worlds including K. Given U a set of possible worlds, $\mathrm{Th}(U)$ returns the associated theory.

A preorder $\preceq$ over W is a reflexive and transitive relation on W. A preorder is *total* if for every $v, w \in W$, either $w \preceq w$ or $w \preceq v$. We take $w \prec v$ as $w \preceq v$ but not $v \preceq w$. A relation $\preceq$ is well founded on W if every non empty subset of W has a non empty subset of $\preceq$-minimal elements.

We will say that a subset X of W is *L-nameable* whenever there exists a formula A in L such that $X = [A]$. When working with relations on W, we will refer to a property that Lewis [14] called the *limit assumption*. A preorder relation R on W satisfies the limit assumption if and only if for any satisfiable formula A in L there exists a set of R-minimal A-words. This requirement is in general weaker than the well foundedness condition. The limit assumption just requires that L-nameable non empty subsets of W have minimal elements, as opposed to requiring so for every subset of W.

2.2 Revision functions

Expansion and revision are the two AGM operations that deal with "accommodating" a new formula into the current theory. While expansion is a simple addition function $+ : I\!\!K \times L \to I\!\!K$ defined as $K + A = \mathrm{Cn}(K \cup \{A\})$, the AGM revision operation has a more subtle definition. The revision function $* : I\!\!K \times L \to I\!\!K$ takes a theory K and a formula A to a revised theory $K * A$, and the eight AGM postulates (K*1)-(K*8) constrain what a revision function can be.

(K*1) $K * A$ is a theory.
(K*2) $A \in K * A$.
(K*3) $K * A \subseteq K + A$.
(K*4) If $\neg A \notin K$ then $K + A \subseteq K * A$.
(K*5) $K * A = \mathrm{Cn}(\bot)$ only if $\mathrm{Cn}(\neg A) = \mathrm{Cn}(\emptyset)$.
(K*6) If $\mathrm{Cn}(A) = \mathrm{Cn}(B)$ then $K * A = K * B$.
(K*7) $K * (A \wedge B) \subseteq (K * A) + B$.
(K*8) If $\neg B \notin K * A$ then $(K * A) + B \subseteq K * (A \wedge B)$.

The first six are called the basic postulates for revision, and they characterize *partial meet revision functions*. Postulates (K*7) and (K*8) are supplementary and they give rise to *transitively relational partial meet revision functions*, which will be our focus of attention. The names of partial meet functions originated in the method for constructing the functions (see [2, 1]). Taken together (K*7) and (K*8) are equivalent to the Ventilation property reported in [1], which provides a factoring on the revision of a theory by a disjunction.

(Ventilation) For all A and B, $K*(A \vee B) = K*A$, or $K*(A \vee B) = K*B$ or $K * (A \vee B) = (K * A) \cap (K * B)$.

A crucial remark about the AGM postulates is that they constrain the behaviour of the functions with respect to all kinds of input sentences but do not deal with varying theories (see [19] and [4]). That is, the postulates indicate nothing about the behaviour of the revision function when applied to different theories $K \in I\!\!K$. In particular we observe that, in general, $*$ is not monotone, in the sense that if one theory is included in another, the revision of the first is not necessarily included in the revision of the second:

$$H \subseteq K \text{ does not imply } H * A \subseteq K * A.$$

Observation 2.1 (follows from [1]) If $*$ is a revision operation satisfying postulates (K*1),(K*4) and (K*5), in a language admitting at least two mutually independent formulae A, B (neither $A \in \mathrm{Cn}(B)$ nor $B \in \mathrm{Cn}(A)$), then monotony fails for $*$.

Proof. Let $K = \mathrm{Cn}(A, B)$, $H_1 = \mathrm{Cn}(A), H_2 = \mathrm{Cn}(B)$. Assume monotony. As $H_i \subseteq K$ for $i \in \{1, 2\}$, by monotony, $H_1 * \neg(A \wedge B) \subseteq K * \neg(A \wedge B)$ and $H_2 * \neg(A \wedge B) \subseteq K * \neg(A \wedge B)$.

By independence, $H_1 = \mathrm{Cn}(A)$ is consistent with $\neg(A \wedge B)$, so by (K*4) $H_1 * \neg(A \wedge B) = \mathrm{Cn}(H_1 \cup \{\neg(A \wedge B)\}) = \mathrm{Cn}(A \wedge \neg B)$.

Likewise, $H_2 * \neg(A \wedge B) = \mathrm{Cn}(H_2 \cup \{\neg(A \wedge B)\}) = \mathrm{Cn}(B \wedge \neg A)$. Hence, both $(A \wedge \neg B)$ and $(B \wedge \neg A)$ are included in $K * \neg(A \wedge B)$. Therefore, by (K*1) $K * \neg(A \wedge B)$ is inconsistent. By postulate (K*5), $\neg(A \wedge B)$ is then inconsistent, contradicting the independence of A and B. QED

Although $* : \mathbb{K} \times L \to \mathbb{K}$ satisfying (K*1)-(K*8) is a function of two arguments the point is that the postulates/partial meet construction tell us very little about relations between $K * A$ and $H * A$ when $K \neq H$; the $*$ operation is in some sense degenerate in its first argument. To understand what is involved we have to look at the class of all partial meet operations. Roughly, if you take any two operations $*_1$ and $*_2$ in the class and define $*_3$ as being like $*_1$ on say K but being like $*_2$ on say $H \neq K$, then $*_3$ is always in the class. David Makinson (personal communication, 1999) has formalized this defining the property of being *essentially right unary*.

Let U, V, X be any sets and let F be a set of functions $f : U \times V \to X$. Let $f_1, f_2 \in F$. We say that a function $g : U \times V \to X$ is a *left mix of two* of f_1 and f_2 iff for all $u \in U$, either: (1) $g(u, v) = f_1(v)$, $\forall v \in V$, or (2) $g(u, v) = f_2(v)$, $\forall v \in V$. A set of functions F is *right unary* iff for all $f_1, f_2 \in F$, every left mix of f_1 and f_2 is in F. In other words, iff F is closed under left mixes. But we are interested in not just in left mixes of two functions but of infinitely many. We need the following

Definition 2.2 Let U, V, X be any sets and let F be a set of functions $f : U \times V \to X$. A function $g : U \times V \to X$ is an (infinitary) *left mix* of F iff for all $u \in U$ there is a function f_u in F such that for all $v \in V$, $g(u, v) = f_u(u, v)$. F is *essentially right unary* (in the infinitary sense) iff every infinitary left mix of F is in F, i.e. iff F is closed under infinitary left mixes.

The desisive observation is that the set of all AGM revision functions is essentially right unary.

Observation 2.3 *The set of functions* $* : \mathbb{K} \times L \to \mathbb{K}$ *satisfying (K*1)-(K*8) is essentially right unary.*

We turn now to a semantic characterization of the AGM revision functions. Among the alternative presentations of the AGM theory, Grove's [9] provides a possible worlds semantics via his constructions based on systems of spheres. A system of spheres S^K for a theory K is a subset of $\mathcal{P}(W)$ containing W, totally ordered under set inclusion, such that $[K]$ is the $\subseteq$-minimal element of S^K. A system S^K should validate the limit assumption: for every satisfiable formula A in the language there exists a $\subseteq$-minimal sphere in S^K (notated as $c^K(A)$) with non-empty intersection with $[A]$.

Definition 2.4 (System of spheres, [9]) A system of spheres S^K centered on theory K is a set of sets of possible worlds that verifies:

(S1) If $U, V \in S$ then $U \subseteq V$ or $V \subseteq U$. (totally ordered)

(S2) For every $U \in S$, $[K] \subseteq U$. (minimum.)

(S3) $W \in S$. (maximum)

(S4) For every $A \in L$ s.t. there is a sphere U in S^K with $[A] \cap U \neq \emptyset$, there is a $\subseteq$-minimal sphere V in S s.t. $[A] \cap V \neq \emptyset$. (limit assumption)

For any sentence A, if $[A]$ has a non-empty intersection with some sphere in S^K then by (S4) there exists a minimal such sphere in S^K, say $c^K(A)$. But, if $[A]$ has an empty intersection with all spheres, then it must be the empty set (since (S3) assures W is in S^K), in this case c^K is put to be just W. Given a system of spheres S^K and a formula A, c^K is defined as:

$$c^K(A) = \begin{cases} W & \text{if } [A] = \emptyset \\ \text{the } \subseteq\text{-minimal sphere } S' \text{ in } S^K \text{ s.t. } S' \cap [A] \neq \emptyset & \text{otherwise.} \end{cases}$$

A system S^K determines a revision function $*$ for K in the sense that for every formula $A \in L$ and every $w \in W$, $w \in [K * A]$ iff $w \in [A]$ and $w \in c^K(A)$.

Observation 2.5 (Grove, [9]) The following are equivalent:

i) The revision operation satisfies (K*1)-(K*8).

ii) There exists a system of spheres S^K such that for all formulae $A \in L$, $K * A = \mathrm{Th}(c^K(A) \cap [A])$.

It is possible to recast a system of spheres centered in $[K]$ as a total preorder $\preceq$ over W, having the elements of $[K]$ as minimal elements, and satisfying the limit assumption (every L-nameable subset of W must have some $\prec$-minimal element). Without loss of generality then a system of spheres centered in $[K]$ can be seen as a function from W to any totally ordered set with smallest element. This set can be taken to be $I\!R^+$, be the set of positive real numbers including 0, but not necessarily so. We define $d_K : W \to I\!R^+$ that decorates with real numbers the nested spheres of a Grove system.

Observation 2.6 For every system of spheres S^K there is a function d_K on $I\!R^+$ such that

$d_K(v) < d_K(w)$ iff $(\exists S_1, S_2 \in S^K)(v \in S_1, w \in S_2$ and $S_1 \subset S_2)$, and
$d_K(v) = d_K(w)$ iff $(\forall S_i \in S^K)(w \in S_i \Leftrightarrow v \in S_i)$.

These functions provide a notion of distance from theories to worlds: If $d_K(w) < d_K(v)$ then w is closer than v or "more consistent" with the current theory K. And this measure can be naturally extended over sets of worlds, by requiring the value assigned to a set X to be the smallest value assigned to the worlds in X. Special consideration is required if X is empty. Let now S^K be any system of spheres and d_K any real function corresponding to it as in Observation 2.6 above. We first extend d_K to any subset of W as follows. Define $d_K : \mathcal{P}(W) \to I\!R^+$ as:

$$d_K(X) = \begin{cases} \min\{d_K(w) : w \in X\} & \text{, if } X \neq \emptyset. \\ 0 & \text{, if } X = \emptyset. \end{cases}$$

In order to represent a system of spheres by a function d_K we should impose the limit assumption on d_K: For every L-nameable $X \subseteq W$, $d_K(X)$ must be defined. If X is not nameable by a single formula then $\{d_K(w) : w \in X\}$ can be a set of infinite descending values where the minimum is not defined.

In the obvious way the function d_K induces a revision function $*$ for a theory K, such that $K * A$ is the theory entailed by the set of A-worlds that are closest to K according to the function d_K. Then, if we take

$$K * A = \text{Th}(\{w \in [A] : d_K(w) = d_K([A])\})$$

the revision operation for K so obtained coincides with the original $*$ operation whose semantic model was S^K. It is important to remark that d_K induces a revision function for a theory K, leaving unspecified the function for all the other theories of $I\!K$.

2.3 Update functions

Katsuno and Mendelzon assume a classical propositional language based on a finite set of propositional variables P. Their simplifying assumption possesses a very convenient consequence: the set of all possible worlds W becomes finite. As a result every subset of W becomes nameable, every theory is finitely axiomatisable by a propositional formula and every total preorder $\preceq$ on W is well founded. The update operator is defined as a binary connective $\diamondsuit$ in the propositional language, which does not have a truth functional behaviour. $B \diamondsuit A$ denotes the result of *updating* the theory $\text{Cn}(B)$ with the formula A.

We generalize the update function to theories, writing the $\diamondsuit$ operator as a binary function that takes a theory and a formula and returns a theory

$\diamondsuit : \mathbb{K} \times L \to \mathbb{K}$. Notice that in a finite propositional language this is just a notational variant of Katsuno and Mendelzon's original setting. The nine postulates governing the update operation are:

(U0) $K\diamondsuit A$ is a theory.
(U1) $A \in K\diamondsuit A$.
(U2) If $A \in K$ then $K\diamondsuit A = K$.
(U3) If $K \neq L$ and A is satisfiable then $K\diamondsuit A \neq L$.
(U4) If $\mathrm{Cn}(A) = \mathrm{Cn}(B)$ then $K\diamondsuit A = K\diamondsuit B$.
(U5) $K\diamondsuit(A \wedge B) \subseteq \mathrm{Cn}(K\diamondsuit A \cup \{B\})$.
(U6) If $B \in K\diamondsuit A$ and $A \in K\diamondsuit B$ then $K\diamondsuit A = K\diamondsuit B$.
(U7) If K is a complete theory then $K\diamondsuit(A \vee B) \subseteq \mathrm{Cn}(K\diamondsuit A \cup K\diamondsuit B)$.
(U8) $\mathrm{Cn}(K \cap K')\diamondsuit A = \mathrm{Cn}((K\diamondsuit A) \cap (K'\diamondsuit A))$.
(U9) If K is complete and $\mathrm{Cn}((K\diamondsuit A)\wedge B) \neq L$ then $\mathrm{Cn}((K\diamondsuit A)\cup\{B\}) \subseteq K\diamondsuit(A \wedge B)$.

Postulate (U8) constranis the update of the intersection of two theories as the intersections of the single updates. There is no analogue of this postulate for revisions. We shall remark that postulate (U9) is optional in Katsuno and Mendelzon's original setting, and gives rise to special updates that we will be our focus of attention.

To semantically characterize the update operation, Katsuno and Mendelzon formalize a notion of closeness between possible worlds. Instead of associating to a theory K a unique ordering (as done in revision), they consider a set of total preorders $\preceq_w$, one for each $w \in W$, such that $v \preceq_w u$ if and only if world v is at least as close to world w as u is. The ordering dictates that the most plausible changes to w to accommodate A lead to those A-worlds that are minimal in $\preceq_w$. They require that each $\preceq_w$ satisfies the following *centering* condition, which says that for every w, no world is as close to w as w itself:

$$\text{If } v \preceq_w w \text{ then } v = w.$$

The following characterization result holds for the update operation (for a finite language).

Observation 2.7 (Katsuno and Mendelzon, [11]) The following are equivalent: i) The update operator $\diamondsuit$ satisfies (U0)-(U9).
ii) There exists an assignment that maps each $w \in W$ to a total preorder $\preceq_w$ such that

$$K\diamondsuit A = \mathrm{Th}\Big(\bigcup_{w\in[K]} \{v \in [A] : v \text{ is } \preceq_w\text{-minimal in}[A]\}\Big)$$

But we are interested in characterization results for the infinite case. The characteristic pointwise semantics of the update function is immediately defined for infinite languages. Only the notion of closeness between worlds requires some adjustment. In addition to the centering condition, each $\preceq_w$ should satisfy the limit assumption: let A be any formula in L, then there exists some non-empty set Y, $Y \subseteq [A]$ such that each element in Y is a $\preceq_w$-minimal element of $[A]$. Formally,

$$\forall w \in W, \forall A \in L, \exists Y \subseteq [A], Y \neq \emptyset \text{ such that } \forall y \in Y, \forall x \in [A], y \preceq_w x.$$

(Notice that the limit assumption is trivially satisfied in finite propositional languages.)

Definition 2.8 (Update function) Let L be a possibly infinite propositional language. Let $\langle W, \{\preceq_w : w \in W\}\rangle$ be such that each $\preceq_w$ is a total preorder over W satisfying the centering condition and the limit assumption. We define $\blacklozenge : I\!\!K \times L \to I\!\!K$ as

$$K \blacklozenge A = \mathrm{Th}(\bigcup_{w \in [K]} \{v \in [A] : v \text{ is } \preceq_w \text{-minimal in } [A]\}).$$

As we reported in [5, 7] postulates (U0)-(U9) do not characterize the update operation in a language with an infinite number of propositional letters.

Observation 2.9 If L is an infinite propositional language, postulates (U0)-(U9) do not fully characterize the $\blacklozenge$ operation.

Proof. Given a propositional language L with an infinite but countable number of propositional letters we will exhibit a function $\circ : I\!\!K \times L \to I\!\!K$ satisfying (U0)-(U9) for which there is no model $\langle W, \{\preceq_w : w \in W\}\rangle$, satisfying that $\forall K \in I\!\!K$, $\forall A \in L, K \circ A = K \blacklozenge A$. We semantically define $\circ$ as follows. Let us single out an (arbitrary) point v in W. For every $K \in I\!\!K$ and for every $A \in L$ define

$$[K \circ A] = \begin{cases} \emptyset & \text{if } [A] = \emptyset. \\ [K] & \text{if } [K] \subseteq [A]. \\ ([K] \cap [A]) \cup \{v\} & \text{if } A \in v \text{ and } [K] \cap [\neg A] \neq \emptyset \text{ is finite.} \\ [A] & \text{if } A \notin v \text{ or } [K] \cap [\neg A] \text{ is an infinite set.} \end{cases}$$

We first check that $\circ$ satisfies postulates (U0)-(U9). By definition $\circ$ trivially satisfies postulates (U0), (U1), (U2), (U3) and (U4).

(U5). We have to show that $K \circ (A \wedge B) \subseteq \mathrm{Cn}(K \circ A \cup \{B\})$ holds. There are three cases.

(a) If $[K] \subseteq [A]$ then $K \circ A = K$. If $\neg B \in K$, then $\mathrm{Cn}(K \circ A \cup \{B\}) = L$ and (U5) is verified. If $\neg B \notin K$, then $\mathrm{Cn}(K \circ A \cup \{B\}) = \mathrm{Cn}(K \cup \{B\})$. Since $A \in K$, $\mathrm{Cn}(K \cup \{B\}) = \mathrm{Cn}(K \cup \{A\} \cup \{B\}) = \mathrm{Cn}(K \cup \{A \wedge B\}) = K \circ (A \wedge B)$. Thus, (U5) holds.

(b) Assume $[K] \cap [\neg A] \neq \emptyset$ is a finite set. If $[K] \cap [\neg A \vee \neg B]$ is an infinite set or $A \wedge B \notin v$ then $K \circ (A \wedge B) = \mathrm{Cn}(A \wedge B)$ and (U5) holds. Suppose $[K] \cap [\neg A \vee \neg B]$ is finite and $A \wedge B \in v$. So $[K \circ (A \wedge B)] = ([K] \cap [A \wedge B]) \cup \{v\}$, while $[K \circ A] = ([K] \cap [A]) \cup \{v\}$. Since $B \in v$, $[K \circ A] \cap [B] = ((([K] \cap [A]) \cup \{v\}) \cap [B]) = ([K] \cap [A] \cap [B]) \cup (\{v\} \cap [B])$ $= ([K] \cap [A] \cap [B]) \cup \{v\} = [K \circ (A \wedge B)]$, thus (U5) is verified.

(c) If $[K] \cap [\neg A]$ is an infinite set then $[K] \cap [\neg A \vee \neg B]$ is also infinite. By definition $[K \circ (A \wedge B)] = [A \wedge B] = \mathrm{Cn}([A] \cup [B]) = \mathrm{Cn}(K \circ A \cup \{B\})$.

(U6). Suppose $B \in K \circ A$ and $A \in K \circ B$.

(a) If $[K] \subseteq [A]$ then $K \circ A = K$. Since $B \in K \circ A$, then $B \in K$, so $K \circ B = K = K \circ A$.

(b) Assume $[K] \cap [\neg A] \neq \emptyset$ is a finite set. If $A \in v$ then $[K \circ A] = ([K] \cap [A]) \cup \{v\}$. Since $B \in K \circ A$, then $([K] \cap [A]) \cup \{v\} \subseteq [B]$, and in particular, $B \in v$. Furthermore $[K] \cap [\neg B] \neq \emptyset$ is finite. Then, by definition, $[K \circ B] = ([K] \cap [B]) \cup \{v\}$. Since, in addition, $A \in K \circ B$, we obtain that $([K] \cap [B]) \cup \{v\} \subseteq [A]$. Therefore, $[K] \cap [A] = [K] \cap [B]$ and hence under the conditions in (b), $K \circ A = K \circ B$. Now suppose $A \notin v$. Then $[K \circ A]$ $= [A]$. Since $B \in K \circ A$, $[A] \subseteq [B]$. As $A \in K \circ B$, $[K \circ B] \subseteq [A]$. Hence $[K \circ B] \neq ([K] \cap [B]) \cup \{v\}$, because we assumed $A \notin v$. Hence, it must be that $[K \circ B] = [B]$, so $[B] \subseteq [A]$. Therefore, $[A] = [B]$ and $K \circ A = K \circ B$.

(c) Assume $[K] \cap [\neg A]$ is an infinite set. Then, $[K \circ A] = [A]$. Since $B \in K \circ A$, then $[A] \subseteq [B]$. There are two possibilities for $K \circ B$. If $[K \circ B] = [B]$ then, using that $A \in K \circ B$, we obtain $[B] \subseteq [A]$ and $[K \circ A] = [K \circ B]$. If $[K \circ B] = ([K] \cap [B]) \cup \{v\}$ then $B \in v$ and $[K] \cap [\neg B]$ is a finite set. Because $A \in K \circ B$, $([K] \cap [B]) \cup \{v\} \subseteq [A]$, and $[K] \cap [B] \subseteq [K] \cap [A]$. Then, $[K] \cap [\neg A] \subseteq [K] \cap [\neg B]$; but this is impossible because we assumed $[K] \cap [\neg A]$ to be an infinite set and $[K] \cap [\neg B]$ to be finite.

(U7). We want to prove that if K is a complete theory then $K \circ (A \vee B) \subseteq \mathrm{Cn}(K \circ A \cup K \circ B)$. Assume K is complete. If $A \in K$, $K \circ A = K$ and $K \circ (A \vee B) = K$. Thus, (U7) holds. If $\neg A \in K$, and $B \in K$, then $K \circ (A \vee B) = K \circ B = K$, so (U7) holds. If $\neg A \in K$, and $\neg B \in K$, if $A \in v$ or $B \in v$, then $K \circ (A \vee B) = v$, and either $K \circ B = v$ or $K \circ A = v$, so (U7) holds. If $\neg A \in v$ and $\neg B \in v$, then we obtain that $K \circ (A \vee B) = \mathrm{Cn}(A \vee B)$, $K \circ B = \mathrm{Cn}(B)$ and $K \circ A = \mathrm{Cn}(A)$. Hence, (U7) is verified.

(U8). We show that $(K_1 \cap K_2) \circ A = (K_1 \circ A) \cap (K_2 \circ A)$. Let $K = K_1 \cap K_2$.

(a) Assume $A \in K$. Then $K_1 \circ A = K_1$, $K_2 \circ A = K_2$ and $K \circ A = K$. Therefore (U8) is validated.

(b) Assume $[K] \cap [\neg A]$ is a finite non-empty set and $A \in v$. Then, $[K \circ A]$ $= ([K] \cap [A]) \cup \{v\}$. If each $[K_i] \cap [\neg A]$, for $i = 1, 2$, is a finite set then $[K_i \circ A]$ $= ([K_i] \cap [A]) \cup \{v\}$, $i = 1, 2$. So $[K \circ A] = ([K_1] \cap [A]) \cup ([K_2] \cap [A]) \cup \{v\} =$

$[K_1 \circ A] \cup [K_2 \circ A]$. Otherwise, suppose $[K_1] \cap [\neg A]$ is an infinite set, and say $A \in K_2$. Then it also holds that $[K_1 \circ A] \cup [K_2 \circ A] = ((([K_1] \cap [A]) \cup \{v\}) \cup [K_2] = ((([K_1] \cap [A]) \cup \{v\}) \cup ([K_2] \cap [A]) = ([K_1] \cap [A]) \cup \{v\} \cup ([K_2] \cap [A]) = ((([K_1] \cup [K_2]) \cap [A]) \cup \{v\} = ([K] \cap [A]) \cup \{v\} = [K \circ A]$.

(c) Assume $[K] \cap [\neg A]$ is an infinite set or $\neg A \in v$. If $\neg A \in v$ then $K \circ A = K_1 \circ A = K_2 \circ A = \mathrm{Cn}(A)$, therefore, (U8) holds. Otherwise, either $[K_1] \cap [\neg A]$ or $[K_2] \cap [\neg A]$ or both are infinite sets. Clearly $[K \circ A] = [A]$ and, say, $[K_1] = [A]$. So $[K \circ A] = [K_1 \circ A]$, therefore, independently of the value of $[K_2 \circ A]$, we obtain that $[K \circ A] = [K_1 \circ A] \cup [K_2 \circ A]$.

(U9). Assume that K is complete and $[K \circ A] \cap [B] \neq \emptyset$. We prove that $[K \circ (A \wedge B)] \subseteq [K \circ A] \cap [B]$.

(a) If $A \in K$, $K \circ A = K$, by the hypotheses, $B \in K$. So $K \circ (A \wedge B) = K$. Thus, (U9) is verified.

(b) If $A \notin K$, then since K is complete $\neg A \in K$. If $A \in v$, $K \circ A = v$. By the hypothesis that $[K \circ A] \cap [B] \neq \emptyset$ we conclude $B \in v$. Thus, $[K \circ (A \wedge B)] \subseteq [K \circ A] \cap [B]$. In fact, $[K \circ (A \wedge B)] = [K \circ A] \cap [B] = \{v\}$. If $A \notin v$, $[K \circ A] = [A]$ and $[K \circ (A \wedge B)] = [A \wedge B]$. Thus, $[K \circ A] \cap [B] = [K \circ (A \wedge B)]$, hence (U9) is verified.

Now suppose for contradiction that there is a model $M = \langle W, \{\preceq_w : w \in W\}\rangle$, where each $\preceq_w$ is a total preorder on W satisfying the limit assumption and the centering condition, such that $\forall K \in \mathbb{K}, \forall A \in L, K \circ A = K \blacklozenge A$. Thus, for every theory K such that $[K]$ is a finite set, and for every formula A, if $\neg A \in K$ and $A \in v$, where v is the distinguished point appearing in the definition of $\circ$ above, $K \circ A = K \blacklozenge A = v$ must hold. This translates into the following condition on the model M.

$$\forall x \in [\neg A], \forall y \in [A], v \neq y, \quad v \prec_x y.$$

Now let K be a theory such that $[K]$ is an infinite set and let $A \in L$ be such that $A \in v$ and $\neg A \in K$. Then by definition of $\circ$, $[K \circ A] = [A]$. However, $[K \blacklozenge A] = \bigcup_{x \in [K]} \{y \in [A] : y \text{ is } \preceq_x\text{-minimal in } [A]\} = \{v\}$. Because the language is infinite $\{v\} \neq [A]$. QED

Since (U0)-(U9) are insufficient to characterize the update operation in an infinite language, we propose the following postulate as a strengthening of Katsuno and Mendelzon's postulate (U8) to achieve the representation result.

(IU8) If $K = \bigcap H_i$ then $K \diamondsuit A = \bigcap (H_i \diamondsuit A)$.

(IU8) states that the update of an intersection is the intersection of the updates. Obviously (IU8) implies (U8). We now prove that postulates (U0)-(U9) plus (IU8) completely characterize the update operation when infinite languages are allowed. We shall remark that Peppas and Williams in [16] have also reformulated the update operation as a function over theories of first order logic and they also proposed the same postulate (IU8). Implicitly, their article claims that Katsuno and Mendelzon's original postulates would be complete for general propositional languages, but not for first order. n our completeness proof we use the following lemma stating the Ventilation condition of [1] for the updates of consistent complete theories.

Lemma 2.10 (Ventilation condition) Let $\diamondsuit$ be an update function satisfying postulates (U0)- (U9). If K is consistent and complete then for all $A, B \in L$, $K\diamondsuit A \vee B = K\diamondsuit A$ or $K\diamondsuit A \vee B = K\diamondsuit A$ or $K\diamondsuit A \vee B = K\diamondsuit A \cap K\diamondsuit B$.

PROOF. Assume $K\diamondsuit A \vee B$ is different from $K\diamondsuit A$ and is also different from $K\diamondsuit B$. We want to prove that $K\diamondsuit A \vee B = K\diamondsuit A \cap K\diamondsuit B$. We will show the double inclusion.

($\supseteq$). This inclusion follows directly from (U5), which requires that $K\diamondsuit A \subseteq \mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\})$ and $K\diamondsuit B \subseteq \mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\})$. Then $K\diamondsuit A \cap K\diamondsuit B \subseteq \mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) \cap \mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\})$. By the rule of introduction of disjunction into the premises, $K\diamondsuit A \cap K\diamondsuit B \subseteq \mathrm{Cn}(K\diamondsuit A \vee B \cup \{A \vee B\}) = K\diamondsuit A \vee B$, using (U0) and (U1).

($\subseteq$). Suppose $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) \neq L$ and $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\}) \neq L$. By (U9) $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) \subseteq K\diamondsuit A$ and $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\}) \subseteq K\diamondsuit B$. Since $K\diamondsuit A \vee B \subseteq \mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) \cap \mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\})$, we have that $K\diamondsuit A \vee B \subseteq K\diamondsuit A \cap K\diamondsuit B$.

Now suppose $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\}) = L$ and $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) \neq L$ (the other is similar). Thus, $\neg B \in K\diamondsuit A \vee B$ and by (U1) $A \in K\diamondsuit A \vee B$. By (U6) If $A \in K\diamondsuit A \vee B$ and $A \vee B \in K\diamondsuit A$ then $K\diamondsuit A \vee B = K\diamondsuit A$, contradicting our initial assumption.

Finally, suppose $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) = L$ and $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\}) = L$. By (U1) $A \vee B \in K\diamondsuit A \vee B$. By $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{A\}) = L$, we have that $\neg A \in K\diamondsuit A \vee B$, thus $B \in K\diamondsuit A \vee B$. But $\mathrm{Cn}(K\diamondsuit A \vee B \cup \{B\}) = L$, so $K\diamondsuit A \vee B = L$. Since K is consistent, by (U3) $A \vee B$ is unsatisfiable. Therefore A, B are both unsatisfiable formulae, and by (U1) $L = K\diamondsuit A \vee B = K\diamondsuit A = K\diamondsuit B$, again contradicting our initial assumptions. QED

Theorem 2.11 Let L be a possibly infinite propositional language, and let Cn be a classical consequence relation that is compact and satisfies the rule of introduction of disjunctions into the premisses. An operator $\diamondsuit$ satisfies postulates (U0)-(U7), (IU8), (U9) if and only if there exists a model $M = \langle W, \{\preceq_w : w \in W\}\rangle$, where each $\preceq_w$ is a total preorder over W centered in w that satisfies the limit assumption and for any $K \in \mathbb{K}$, $A \in L$, $K\diamondsuit A = K\blacklozenge A$.

PROOF. [$\Leftarrow$]. We have to show that the operator $\blacklozenge$ satisfies postulates (U0)-(U7), (IU8) and (U9).

(U0) and (U1). Granted since, by Definition 2.8, $[K\blacklozenge A] \subseteq [A]$.

(U2). Follows as a consequence of the centering condition.

(U3). Follows by the definition of min on nonempty sets.

(U4). Obvious from the semantic definition of the update operation.

(U5). We have to show that $[K\blacklozenge A]\cap[B] \subseteq [K\blacklozenge(A\wedge B)]$. If $[K\blacklozenge A]\cap[B] = \emptyset$, the inclusion trivially holds. Assume $[K\blacklozenge A] \cap [B] \neq \emptyset$. Let u be any in $[K\blacklozenge A] \cap [B]$. Then $u \in \bigcup_{w\in[K]}\{v \in [A] : v \text{ is } \preceq_w\text{-minimal in } [A]\} \cap [B] = \bigcup_{w\in[K]}\{v \in [A] \cap [B] : v \text{ is } \preceq_w\text{-minimal in } [A]\}$. Let $w_0 \in [K]$ be such that u is $\preceq_{w_0}$-minimal in $[A]$. That is $\forall v \in [A]$, $u \preceq_{w_0} v$. A fortiori, $u \in [A]\cap[B]$. Thus, there is no $v \in [A]\cap[B]$ such that $v \prec_{w_0} u$, so u is indeed $\preceq_w$-minimal in $[A] \cap [B]$.

(U6). Assume $B \in K\blacklozenge A$ and $A \in K\blacklozenge B$. We want to show $[K\blacklozenge A] = [K\blacklozenge B]$. $[K\blacklozenge A] = \bigcup_{w\in[K]}\{v \in [A] : v \text{ is } \preceq_w\text{-minimal in } [A]\}$. By the hypothesis that $B \in K\blacklozenge A$, $[K\blacklozenge A] \subseteq [B]$. $[K\blacklozenge A] = \bigcup_{w\in[K]}\{v \in [A] \cap [B] : v \text{ is } \preceq_w \text{-minimal in } [A] \cap [B]\}$. Similarly, $[K\blacklozenge B] = \bigcup_{w\in[K]}\{v \in [B] : v \text{ is } \preceq_w\text{-minimal in } [B]\}$, and by the hypothesis that $A \in K\blacklozenge B$, $[K\blacklozenge B] \subseteq [A]$. $[K\blacklozenge B] = \bigcup_{w\in[K]}\{v \in [A] \cap [B] : v \text{ is } \preceq_w \text{-minimal in } [A] \cap [B]\}$. Therefore, $[K\blacklozenge A] = [K\blacklozenge B]$, as required.

(U7). We have to prove that when $[K]$ is a singleton $[K\blacklozenge A] \cap [K\blacklozenge B] \subseteq [K\blacklozenge(A\vee B)]$. Assume $[K] = \{u\}$. Then, $[K\blacklozenge A] = \{v \in [A] : v \text{ is } \preceq_u\text{-minimal in } [A]\}$, while $[K\blacklozenge B] = \{v \in [B] : v \text{ is } \preceq_u\text{-minimal in } [B]\}$. Furthermore $[K\blacklozenge(A \vee B)] = \{v \in [A \vee B] : v \text{ is } \preceq_u\text{-minimal in } [A \vee B]\} = \{v \in [A] \cup [B] : v \text{ is } \preceq_u\text{-minimal in } [A] \cup [B]\} = \{v \in [A] \cup [B] : v \text{ is } \preceq_u\text{-minimal in } [A] \text{ or } v \text{ is } \preceq_u\text{-minimal in } [B]\}$. And finally, $[K\blacklozenge A] \cap [K\blacklozenge B] = \{v \in [A] \cap [B] : v \text{ is } \preceq_u \text{-minimal in } [A] \text{ and } v \text{ is } \preceq_u \text{-minimal in } [B]\}$. Thus, $[K\blacklozenge A] \cap [K\blacklozenge B] \subseteq [K\blacklozenge(A \vee B)]$.

(IU8). Assume $[K] = \bigcup_{i\in I}[K_i]$ to show $[K\blacklozenge A] = \bigcup_{i\in I}[K_i\blacklozenge A]$. By definition, $[K\blacklozenge A] = \bigcup_{w\in\bigcup_{i\in I}[K_i]}\{v \in [A] : v \text{ is } \preceq_w\text{-minimal in } [A]\} = \bigcup_{i\in I}(\bigcup_{w\in[K_i]}\{v \in [A] : v \text{ is } \preceq_w \text{-minimal in } [A]\}) = \bigcup_{i\in I}[K_i\blacklozenge A]$.

(U9). Assume $[K] = \{u\}$ and $([K\blacklozenge A]) \cap [B] \neq \emptyset$. We have to show $[K\blacklozenge(A \wedge B)] \subseteq [K\blacklozenge A] \cap [B]$. Suppose there is some $y \in [K\blacklozenge A \wedge B]$ but $y \notin [K\blacklozenge A] \cap [B]$. Then $[K\blacklozenge A] \subseteq [\neg B]$, contradicting $[K\blacklozenge A] \cap [B] \neq \emptyset$.

[$\Rightarrow$]. Let $\diamondsuit$ be a change function satisfying (U0)-(U7), (IU8) and (U9). We will construct a model $M = \langle W, \{\preceq_w : w \in W\}\rangle$ such that for every theory $K \in I\!\!K$ and formula $A \in L$, $K\diamondsuit A = K\blacklozenge A$.

We start by defining the model M. The domain W will be the set of all complete consistent theories in the language L. Assume $\{\preceq_w : w \in W\}$ is the set of relations defined by:

(i.) $v \preceq_w u$ iff there exists $A \in v \cap u$ such that $v \in [w\diamondsuit A]$ or there exists no satisfiable A such that $u \in [w\diamondsuit A]$.

We will show that M is an update model by demonstrating that each $\preceq_w$ is a total preorder satisfying the centering condition and the limit assumption.

(a) $\preceq_w$ is *totally connected*. Suppose $u \not\preceq_w v$ and $v \not\preceq_w u$. Then for some consistent $A, B \in L$, $v \in [w\diamondsuit A]$ and $u \in [w\diamondsuit B]$. Then, by Lemma 2.10 we have that $w\diamondsuit A \vee B = w\diamondsuit A$ or $w\diamondsuit A \vee B = w\diamondsuit B$ or $w\diamondsuit A \vee B = w\diamondsuit A \cap w\diamondsuit B$. Thus one of v or u is in $[w\diamondsuit A \vee B]$ contradicting the fact that neither $v \preceq_w u$ nor $u \preceq_w v$. (Notice that total connectedness implies reflexivity).

(b) $\preceq_w$ is *transitive*. Suppose $u \preceq_w v$ and $v \preceq_w z$. If there is no satisfiable A such that $z \in [w\diamondsuit A]$ then also $u \preceq_w z$ and we are done. Otherwise, $v \preceq_w z$ because there exists $C \in u \cap v$ such that $u \in [w\diamondsuit C]$, but then also there exists $B \in v \cap z$ such that $v \in [w\diamondsuit B]$. Now $\neg C \notin K\diamondsuit B$, so by Lemma 2.10 $\neg C \notin [w\diamondsuit B \vee C]$. This means that $w\diamondsuit((B \vee C) \wedge C)$ $=w\diamondsuit(B \vee C) \cup \{C\}$ $= K\diamondsuit C$. Namely $[w\diamondsuit C] = [w\diamondsuit B \vee C] \cap [C]$. Then $[K\diamondsuit C] \subseteq [K\diamondsuit(B \vee C)]$ and $u \in [K\diamondsuit(B \vee C)]$. But $B \vee C \in z \cap u$, so $u \preceq_w z$.

(c) $\preceq_w$ is *centered*. Suppose $v \neq w$ and $v \preceq_w w$. Trivially, from the postulates, $w \in [w\diamondsuit\top]$, hence by definition (i.) $v \preceq_w w$ implies there is some $A \in v \cap w$ such that $v \in [w\diamondsuit A]$. But this contradicts postulate (U2) which requires $[w\diamondsuit A] = \{w\}$.

(d) That $\preceq_w$ satisfies the *limit assumption* follows directly from postulate (U3), which implies that for every satisfiable A, and for every $w \in W$, $[w\diamondsuit A]$ must be non empty. Then there must be some $v \in [A]$ that is minimal in $\preceq_w$ such that $v \in [w\diamondsuit A]$.

It remains to show that the update function determined by M is $\diamondsuit$. In the limiting case when K is the inconsistent theory or A is unsatisfiable, $K\diamondsuit A$ and $K\blacklozenge A$ agree. We will now prove, for K and A satisfiable, that $u \in [K\diamondsuit A]$ iff $u \in [K\blacklozenge A]$ by analyzing the different cases.
Suppose $[K] = \{w\}$.

$[K\diamondsuit A] \subseteq [K\blacklozenge A]$. Let $v \in [K\diamondsuit A]$. By postulate (U1), $[K\diamondsuit A] \subseteq [A]$, so $v \in [A]$. By (i.), for every $u \in [A]$, $v \preceq_w u$. Hence, $v \in \{y \in [A] :$ y is $\preceq_w$-minimal in $[A]\}= [K\blacklozenge A]$.

$[K\blacklozenge A] \subseteq [K\diamondsuit A]$. Let $v \in [K\blacklozenge A]$. By definition of $\blacklozenge$, $v \in \{y \in [A] :$ y is $\preceq_w$-minimal in $[A]\}$. So for all $u \in [A]$, $v \preceq_w u$; thus, by (i.), $v \in [w\diamondsuit A]$.
The general case, $[K] > 1$.

$[K\diamondsuit A] \subseteq [K\blacklozenge A]$. Let $v \in [K\diamondsuit A]$. By postulate (IU8), if $[K] = \bigcup_{i\in I}[K_i]$ then $[K\diamondsuit A] = \bigcup_{i\in I}[K_i\diamondsuit A]$.

In particular, $[K] = \bigcup_{i\in I}[T_i]$ for complete theories T_i. Thus, $v \in \bigcup_{i\in I}[T_i\diamondsuit A]$. Hence, v must be in, say, some $[T_j\diamondsuit A]$, $j \in I$. Then, by the previous case, $v \in [T_j\blacklozenge A]$. Therefore, $v \in \bigcup_{w\in[K]}\{y \in [A] : y$ is $\preceq_w$ -minimal in $[A]\}= [K\blacklozenge A]$.

$[K\blacklozenge A] \subseteq [K\diamondsuit A]$. Let $v \in [K\blacklozenge A]$. Then, $v \in \bigcup_{w\in[K]}\{y \in [A] : y \text{ is } \preceq_w\text{-minimal in } [A]\}$. In particular, there exists some $w \in [K]$ such that $v \in \{y \in [A] : y \text{ is } \preceq_w \text{-minimal in } [A]\}$. By the previous case, $v \in [w\diamondsuit A]$. But $[K] = \bigcup_{i\in I}[T_i]$ for complete theories T_i, such that $w = T_j$, for some $j \in I$. By postulate (IU8) we obtain that when $[K] = \bigcup_{i\in I}[K_i]$, $[K\diamondsuit A] = [T_j\diamondsuit A] \cup (\bigcup_{i\in I, i\neq j}[T_i\diamondsuit A])$. Hence $v \in [K\diamondsuit A]$. QED

Katsuno and Mendelzon's characterization results based on partial orders as opposed to partial pre-orders also lift to the infinite case, replacing postulate (U8) with postulate (IU8).

3 Analytic revision functions

Our aim is to define the AGM revision function in the pointwise semantic framework of update. Consider a theory as a set of possible scenarios. Katsuno and Mendelzon's operation can be calculated by means of a case analysis over the set of complete scenarios compatible with the original theory. First, for each case find out its closest outcome that accommodates the new information; then take as the overall result what is common to all outcomes. Even though for each case the closest outcome entailing the new information is selected, some outcomes could be relatively implausible. Could we have a measure to determine when one outcome is more plausible than another? We suggest that one outcome is more plausible than another when it is at a *closer distance* from the theory under change. We will first formalize a notion of distance and then define a new operation that picks as a result of the change just the outcomes that are minimally distant. We will call this operation an *analytic revision*.

A *distance* is a binary function $f : X \times X \to Y$, such that X is a set and Y is a totally ordered set with minimal element, satisfying that $f(x, y) = \min(Y)$ iff $x = y$ (centering) and $f(x, y) = f(y, x)$ (symmetry). But there are weaker notions. Ultrametric distances satisfy the centering and the triangular inequality and pseudo distances just satisfy the centering condition. Since we seek the connection between revision and update, we are interested in a notion of distance that corresponds to the preorders of update models. Thus, we shall be concerned with pseudo distances only, and making some language abuse we will refer to them just as distances.

It is possible to recast an update model $M = \langle W, \{\preceq_w : w \in W\}\rangle$ into a model based on functions having as range any totally ordered set with smallest element. We will consider the set $\mathbb{R}^+$ of real numbers greater or equal to 0, but any other totally ordered set with smallest element would do. It is clear how each total preorder in the update model induces a function d_w such that all the information encoded in $\preceq_w$ is placed in $d_w : W \to \mathbb{R}$.

$$v \preceq_w u \text{ iff } d_w(v) \leq d_w(u).$$

The centering condition establishes a restriction on the possible values of the functions.

(centering) $d_w(w) = 0$ and for every $v \in W$ such that $v \neq w$, $d_w(v) > 0$.

For each indexical total preorder $\preceq_w$ the limit assumption requires that for each L-nameable set $[A]$ there exists some $\preceq_w$-minimal elements of $[A]$.

(limit assumption) For each $x \in W$, for each $[A] \subseteq W$, there are $y \in [A]$ such that $\forall y' \in [A]$, $d_x(y) \leq d_x(y')$.

The update of K by A is defined as:

$$K \Diamond A = \mathrm{Th}\left(\bigcup_{w \in [K]} \{v \in [A] : d_w(v) = d_w([A])\}\right).$$

Since functions d_w obey the centering condition, the distance from a point to itself is 0 and the distance from a point to every other point is greater than 0. We will require no further properties on d for the moment. Notice in particular that this conception of distance is not symmetric since $d(w, v)$ may differ from $d(v, w)$. Boutilier (personal communication) has provided a good rational for it: "The lack of symmetry seems certainly appropriate when the ordering mirrors exogenous change; for instance, it is quite easy to break an egg while it is hopeless to put it back together." Just for convenience we give the following

Definition 3.1 (Distance between two points) Let an update model $M = \langle W, \{d_w : w \in W\}\rangle$ be given. We define the *distance* function $d : W \times W \to I\!R^+$ between pairs of worlds v, w as the value of w in d_v: $d(v, w) = d_v(w)$.

We shall extend the above definition to distance between sets, as the result of a double minimization. The definition of $d : \mathcal{P}(W) \times \mathcal{P}(W) \to I\!R^+$ covers the limiting case of the empty proposition in a way that will be convenient.

Definition 3.2 (Distance from a set to a set) Let d be a distance function obtained from an update model $\langle W, \{d_w : w \in W\}\rangle$. Let X, Y be subsets of W. Let $f : W \to I\!R^+$ be any positive (greater than 0) function. We define.

$$d(X, Y) = \begin{cases} \min_{x \in X} \min_{y \in Y} \{d(x, y)\} & , \text{ if } X, Y \neq \emptyset. \\ \min_{y \in Y} \{f(y)\} & , \text{ if } X = \emptyset, Y \neq \emptyset. \\ 0 & , \text{ if } Y = \emptyset. \end{cases}$$

¿From now on we assume the extended distance function and, abusing notation, we will write singleton sets without braces, i.e. we will write $d(u,v)$ instead of $d(\{u\},\{v\})$. As before, notice the lack of symmetry: in general $d(X,Y)$ is different from $d(Y,X)$. Furthermore we will directly consider models $M = \langle W, d\rangle$ instead of the indexical models as we can straightforwardly move from one to the other. We are ready now to give the formal semantic definition of analytic revision.

Definition 3.3 (Analytic revision) Let $M = \langle W, d\rangle$ and $X, Y \subseteq W$, then the analytic revision $\bullet : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$ is defined as

$$X \bullet Y = \{y \in Y : d(X,y) = d(X,Y)\}.$$

The syntactic counterpart taking as arguments a theory and a formula, $\bar{\bullet} : I\!\!K \times L \to I\!\!K$ is simply

$$K \bar{\bullet} A = \mathrm{Th}([K] \bullet [A]).$$

4 Connections

4.1 Analytic revision and update

The crucial semantic difference between analytic revision and update is that analytic operation relies on two minimizations while the update just one. As a direct consequence an analytic revision ignores some of the possible outcomes that an update would consider. Then the theory resulting from an analytic revision is at least as informed as that of an update.

Observation 4.1 If K is consistent, $K \diamondsuit A \subseteq K \bar{\bullet} A$.

Proof. We want to show that $X \bullet Y$ is included in $X \diamondsuit Y$. Suppose $y \in X \bullet Y$. Then $\min_{x\in X}\{d(x,y)\} = \min_{x\in X, y'\in Y}\{d(x,y')\}$. Fix a value x_0 of $x \in X$ such that $d(x_0,y) = \min_{x\in X}\{d(x,y)\}$. Then $d(x_0,y) = \min_{x\in X, y'\in Y}\{d(x,y'\}$. Hence $d(x_0,y) = \min_{y'\in Y}\{d(x_0,y'\}$. Hence $y \in X \diamondsuit Y$. QED

The reason for this observation being relative to the consistency of K is that the update function of the inconsistent theory results in the inconsistent theory. In contrast, analytic revision overcomes inconsistency. The following result asserts that when the theory is also complete the two operations coincide.

Observation 4.2 If K is consistent and complete then $K \bar{\bullet} A = K \diamondsuit A$.

PROOF. The proof is quite trivial. Let K be consistent and complete, so its proposition is a singleton $[K] = \{u\}$.

Then, $[K\diamondsuit A] = \bigcup_{i\in[K]}\{w \in [A] : d_i(w) = d_i([A])\} = \{w \in [A] : d_u(w) = d_u([A])\} = \{w \in [A] : d([K], w) = d([K], [A])\} = [K] \bullet [A]$. QED

We establish precisely the connection between analytic revision and update, generalizing the two results above.

Observation 4.3 Let K be a consistent theory and $\langle W, d\rangle$ a structure for the update operation $\diamondsuit$. Then for every formula A there exists a consistent theory $K' \supseteq K$ such that $K\bar{\bullet}A = K'\diamondsuit A$. In particular, K' may be chosen as $\mathrm{Th}(\{w \in [K] : d(w, [A]) = d([K], [A])\})$. (Notice that K' depends on A.)

PROOF. By Observation 4.1 we know that taking $K' = K$ provides us with a theory that is too weak to satisfy the observation. Let's study this in detail.

If A is a satisfiable formula, $[A] \neq \emptyset$, so $[K\diamondsuit A]$ is not empty.

By definition $[K\diamondsuit A] = \bigcup_{w\in[K]}\{v \in [A] : d_w(v) = d_w([A])\} = \bigcup_{w\in[K]}\{v \in [A] : d(w, v) = d(w, [A])\} =$

$\bigcup\{\{v \in [A] : d(w, v) = d(w, [A])\} : w \in [K]$ and $d(w, [A]) = d([K], [A])\}$

$\cup$

$\bigcup\{\{v \in [A] : d(w, v) = d(w, [A])\} : w \in [K]$ and $d(w, [A]) > d([K], [A])\}$.

Thus, $[K']$ should be chosen as $[K'] = \{w \in [K] : d(w, [A]) = d([K], [A])\}$ in which case $K'\diamondsuit A = K\bar{\bullet}A$. QED

The next lemma states that when a formula is consistent with the theory, the analytic revision operation is just the addition of the formula to the theory.

Lemma 4.4 If A is consistent with K, then $K\bar{\bullet}A = \mathrm{Cn}(K \cup \{A\})$.

PROOF. Assume A is consistent with K. Then $[K] \cap [A] \neq \emptyset$. By the centering condition $d([K], [A]) = 0$ and for any $v \notin [K], d([K], v) > 0$. Then by Definition 3.3, $[K] \bullet [A] = \{w \in [A] : d([K], w) = 0\}$. Thus, $[K] \bullet [A] = [K] \cap [A]$. QED

In spite of the technical connection it is not surprising to find that the analytic revision is not an update operator.

Observation 4.5 $\bar{\bullet}$ satisfies (U0)-(U7) and (U9), fails (IU8) and fails monotony.

PROOF. Let's see first that $\bar{\bullet}$ satisfies (U1)-(U7) and (U9).

(U0) and (U1) are granted since by Definition 3.3, $[K] \bullet [A] \subseteq [A]$.

(U2) follows as a direct consequence of Lemma 4.4.

(U3) is a consequence of the limit assumption of d.

(U4) is obvious from the semantic definition of analytic revision.

(U5)[1] We have to show that $([K] \bullet [A]) \cap [B] \subseteq [K] \bullet [A \wedge B]$. If $([K] \bullet [A]) \cap [B] = \emptyset$, the inclusion trivially holds.

Assume $([K] \bullet [A]) \cap [B] \neq \emptyset$. By Definition 3.3, $([K] \bullet [A]) \cap [B] = \{w \in [A] : d([K], w) = d([K], [A])\} \cap [B] = \{w \in [A] \cap [B] : d([K], w) = d([K], [A])\}$. Also, $[K] \bullet [(A \wedge B)] = \{w \in [A] \cap [B] : d([K], w) = d([K], [A] \cap [B])\}$.

Suppose for contradiction that (1) $u \in ([K] \bullet [A]) \cap [B]$, and (2) $u \notin [K] \bullet [(A \wedge B)]$. From (1) we obtain (3) $u \in [A] \cap [B]$, while (2) can be rewritten as (2') $u \notin \{w \in [A] \cap [B] : d([K], w) = d([K], [A] \cap [B])\}$.

Then by (2') and (3) we obtain (4) $d([K], u) > d([K], [A] \cap [B])$. By (1) we have that $d([K], u) = d([K], [A])$, and (3) assures that $u \in [A] \cap [B]$. Hence we obtain $d([K], u) = d([K], [A] \cap [B])$, contradicting (4).

(U6) Assume $B \in K \bar{\bullet} A$ and $A \in K \bar{\bullet} B$.
Since $d([K], [A]) = \min_{x \in [K]} \min_{y \in [A]} \{d(x, y)\}$, there exists $v \in [K] \bullet [A]$ such that $d([K], v) = d([K], [A])$. Similarly, there exists $w \in [K] \bullet [B]$ such that $d([K], w) = d([K], [B])$.

Since $[K] \bullet [A] \subseteq [B]$, then $d([K], [A]) = d([K], v) \geq d([K], [B])$. Also since $[K] \bullet [B] \subseteq [A]$ $d([K], [B]) = d([K], w) \geq d([K], [A])$. We obtain $d([K], [A]) \geq d([K], [B]) \geq d([K], [A])$, thus, $d([K], [A]) = d([K], [B])$. We conclude, $[K] \bullet [A] = [K] \bullet [B]$, as required.

(U7) Assume $[K] = \{u\}$, then distance from $[K]$ is exactly distance from u and $d(u, [A \vee B]) = d(u, [A] \cup [B]) = \min\{d(u, [A]), d(u, [B])\}$. Without loss of generality assume $d(u, [A]) \leq d(u, [B])$. Then $[K] \bullet [A \vee B] = [K] \bullet [A]$; hence, $([K] \bullet [A]) \cap ([K] \bullet [B]) \subseteq [K] \bullet [A \vee B]$.

(U9)[2] We have to show that if $[K]$ is a singleton and $([K] \bullet [A]) \cap [B]$ is not empty then $([K] \bullet [A \wedge B]) \subseteq ([K] \bullet [A]) \cap [B]$. Assume (1) $([K] \bullet [A]) \cap [B] \neq \emptyset$. Then there is some $x \in [A] \cap [B]$ such that $d([K], [A]) = d([K], x)$.

Suppose (2) $[K] \bullet [A \wedge B] \not\subseteq ([K] \bullet [A]) \cap [B]$. Then there is some $u \in [K] \bullet [A \wedge B)])$ but $u \notin ([K] \bullet [A]) \cap [B]$. By (1) and (2) we obtain (3) $d([K], [A \wedge B]) = d([K], u) > d([K], [A])$. By Definition 3.3 and (3), for every $w \in [A]$, if $d([K], w) = ([K], [A])$ then $w \in [A] \cup [\neg B]$, contradicting (1). Notice for later use that for this proof we have not made use of the hypothesis that $[K]$ is a singleton.

To prove that $\bar{\bullet}$ fails postulate (IU8) suffices to to provide witnesses to $(X \cup Y) \bullet Z \neq (X \bullet Z) \cup (Y \bullet Z)$. Let $X, Y, Z \subseteq W$ non-empty, such that $X \cap Z = \emptyset$ and $Y \cap Z \neq \emptyset$. Hence $(X \cup Y) \cap Z = Y \cap Z \neq \emptyset$.

By Lemma 4.4,$Y \bullet Z = Y \cap Z$ and $(X \cup Y) \bullet Z = (X \cup Y) \cap Z = Y \cap Z$. Therefore, $(X \cup Y) \bullet Z = Y \bullet Z$. From postulate (U3) proved above, $X \bullet Z \neq \emptyset$.

Since $X \bullet Z$ may not be included in $Y \bullet Z$, (U8) may not be satisfied. For instance let $X = \{x\} \subseteq [A \wedge \neg B]$, $Y = \{y\} \subseteq [A \wedge B \wedge C]$, $Z = [B]$,

[1] Notice that this postulate corresponds to the AGM revision postulate (K*7).

[2] Notice that this postulate is a particular case of the AGM revision postulate (K*8).

and let $v \in [B \wedge \neg C]$. Let d_x, d_y satisfy the centering condition such that, $d_x(v) = 1$. Then, $v \in X \bullet Z$ and $Y \bullet Z = \{y\}$. Thus, $(X \cup Y) \bullet Z$ is different from $(X \bullet Z) \cup (Y \bullet Z)$.

That $\bar{\bullet}$ fails monotony can be proved using the same strategy of Observation 2.1. QED

The analytic revision operation relies only on those possible worlds that regard the change as minimally distant from the theory under change. Then, if possible, the analytic revision will understand new information as having caused no change at all, a mere confirmation of what already was a possibility in our picture of the world. This behaviour has been stated as Lemma 4.4 and is shared with AGM revision. In the next section we will show that AGM revisions and analytic revisions are indeed connected.

4.2 Analytic revision and AGM revision

First we will note that the analytic revision function $\bar{\bullet}$ satisfies the AGM postulates (K*1)-(K*8).

Theorem 4.6 $\bar{\bullet}$ is a revision operator satisfying (K*1)-(K*8).

Proof. Most postulates follow directly from Definition 3.3 or from Lemma 4.4. (K*7) and (K*8) have been proved as postulates (U5) and (U9) respectively, in Observation 4.5. QED

The key idea behind an analytic revision is to define a meaningful distance relation between sets in terms of the functions d_w (which in turn were obtained from the ternary relations $\preceq_w$). For example, a candidate distance from a theory K could have been any arbitrary d_v. But it is evident that the change operation this approach would induce does not satisfy the complete set of AGM revision postulates.

Observation 4.7 Assume L a language with at least two propositional letters, K an incomplete theory of L, $v \in [K]$ a single element of W and d_v an real function for v satisfying the centering condition. Let $\circ$ be a change operation for K defined as $K \circ A = \mathrm{Th}(\{y \in A : d_v(y) = d_v([A])\})$. Then $\circ$ satisfies (K*1),(K*2),(K*5)-(K*8) but in general fails (K*3)(K*4).

Proof. (K*1),(K*2),(K*5)-(K*8) have identical proofs as those in Theorem 4.6.

(K*3). Since we assume K is not complete then there is a formula A such that $A, \neg A \notin K$. Then, either $v \in [A]$ or $v \in [\neg A]$. Without loss of generality, suppose $v \in [\neg A]$. Then, there is some $x \in [K] \cap [A]$. We show a counterexample to (K*3) such that $x \notin [K \circ A]$. Since L has at least two propositional letters, there is some $u \in [A]$, $u \neq x$. Let $d_v(u) < d_v(x)$. Then,

$x \notin [K \circ A] = \{y \in [A] : d_v(y) = d_v([A])\}$, as x is not a minimal element in d_v satisfying A.

If we add to the the previous counterexample that $u \notin [K]$ and $d_v(u) = d_v([A])$, then postulate (K*4) also fails as $u \in [K \circ A]$ but $u \notin [K] \cap [A]$. QED

Distance from theory K becomes the standard ordering used in the semantic presentations of AGM revision (a world w is as close as v from theory K if and only if the distance from $[K]$ to w is not greater than the distance from $[K]$ to v).

Theorem 4.6 showed that every analytic revision function is an AGM revision function. However, what is most interesting is that *a transitively relational AGM revision function for a theory K is an analytic revision for such theory K*. Only after this result we can speak of a true connection between AGM revision and the semantic structure of update.

Theorem 4.8 (Makinson, personal communication) Every revision function $*$ for K satisfying the extended set of AGM postulates (K*1)-(K*8) is an analytic revision function for K.

PROOF. Let $*$ be an AGM revision function for K satisfying (K*1)-(K*8). By Grove's result, there is a system of spheres S^K for K that represents $*$. By Observation 2.6 S^K induces a real function d_K on W into the reals greater or equal 0, satisfying (centering) and (limit assumption).

The proof of the theorem just consists in showing that *any* real function $d : W \to I\!R^+$ satisfying (centering) and (limit assumption) can be extended to a distance function, obtaining the semantic structure of analytic revision. We define $d : W \times W \to I\!R^+$ as follows.

i. $\forall w, v \in [K], w \neq v$, $d(w, v) = 1$,
ii. $\forall w \in W$, $d(w, w) = 0$,
iii. $\forall w \in [K], \forall v \in W \setminus [K]$, $d(w, v) = d_K(v)$,
iv. $\forall w \in W \setminus [K], \forall v \in W, d(w, v) = g_w(v)$,

where $g_w : W \to I\!R^+$ is any function at all assigning values greater than 0. We extend d as a function on sets as usual, taking $d(\emptyset, v) = d_K(v)$, for the empty set. We have to check that the function d is of the kind needed to generate a analytic revision operation. We just check that the induced relations $\preceq_w$ over W defined by setting

$$u \preceq_w v \text{ iff } d(w, u) \leq d(w, v), \text{ for all } u, v, \in W$$

satisfy (1) $\preceq_w$ is a a total preorder on W, and (2) $\preceq_w$ is centered at w; i.e. if $v \preceq_w w$ then $v = w$.

Now (1) is immediate. To prove (2), let $u \in W$ with $u \neq w$. We want to show that $w \prec_w u$; i.e. that if $u \neq w$ then $d(w, w) < d(w, u)$. By the second case of our definition of d, $d(w,w) = 0$ for all $w \in W$, hence we have to show that for $w \neq u$, $d(w,u) > 0$. If u, w are both in $[K]$, then by the first case $d(w,u) = 1 > 0$. If w is not in $[K]$, it follows from the fourth case that $d(w,u) > 0$. If w in $[K]$ and u is not, then $d(w,u) = r(u) > 0$ since r is itself centered in $[K]$. Thus in all cases $d(w,u) > 0$ and we are done.

It is immediate from the definition of d that (3) for all $u, v \in W \setminus [K]$, for any $w \in [K]$, $d(w,v) \leq d(w,u)$ iff $d_K(v) \leq d_K(u)$ iff $c^K(v) \subseteq c^K(u)$, and (4) for any $u, w \in [K]$ and for all $v \in W \setminus [K]$, $d(w,v) = d(u,v) = d([K],v) = d_K(v)$.

Now let $\bullet$ be the analytic revision function determined by the structure $\langle W, d\rangle$. We have to show that for all A, $[K * A] = [K] \bullet [A]$. If $[K] \cap [A] \neq \emptyset$, by (K*4) in Lemma 4.4 we have $[K * A] = [K] \cap [A] = [K] \bullet [A]$. Suppose $[K] \cap [A] = \emptyset$. By definition of analytic revision and (4) $[K] \bullet [A] = \{v \in [A] : d([K], v) = d([K], [A])\} = \{v \in [A] : d_K(v)$ is $\leq$-minimal in $\{d_K(w) : w \in [A]\}\} = \{v \in [A] : v$ is in the $\subseteq$-minimal sphere in S that intersects $[A]\} =$ (by (3) above) $[K * A]$. QED

We observe in the proof above and also in Definition 3.2 that we have considerable freedom when defining the behaviour of the revision for the inconsistent theory. For example we could require what Makinson called the Overkilling property (O). It says that the analytic revision of an inconsistent theory should result in plain acceptance of the new information.

$$\text{(O) If } K \text{ is inconsistent then } K \bar{\bullet} A = \mathrm{Cn}(A).$$

Coincidentally, this property defines the revision of the inconsistent theory in [13]. The analytic revision function that comply with (O) can be characterized by the function $f : W \to I\!R^+$ involved in the definition of d (see Definition 3.2).

Observation 4.9 (Makinson, personal communication) $\bar{\bullet}$ satisfies (K*1)-(K*8) and (O) if and only if f is a constant function.

PROOF. $\bar{\bullet}$ satisfies (K*1)-(K*8) and (O) iff, by Theorem 4.6 and 4.8, $\bar{\bullet}$ is a analytic revision in $\langle W, d\rangle$ s.t. if K is inconsistent then $K \bar{\bullet} A = \mathrm{Cn}(A)$ iff $\bar{\bullet}$ is a analytic revision in $\langle W, d\rangle$ and for any A, $\{w \in [A] : d(\emptyset, w) = d(\emptyset, [A])\} = [A]$. Now, $\{w \in [A] : d(\emptyset, w) = d(\emptyset, [A])\} = [A]$ iff for any v, w in $[A]$, $f(w)= f(v)$ iff f is a constant function. QED

5 Representation Theorems

Theorem 4.8 proved the correspondence between analytic revisions and AGM transitively relational partial meet revisions *of a given theory*. However, analytic revisions of distinct theories are delicately balanced, while AGM revisions of distinct theories can be totally independent. That is, the set of

AGM revisions is essentially right unary (Observation 2.3), while the set of analytic revisions is not. Can we impose additional constraints to the set of AGM revision functions to achieve the needed balance corresponding to analytic operations? We are looking for postulates that link the behaviour of AGM revision of different theories. In the case of a finite propositional language the needed postulate is dual to the Ventilation condition, which we name

(K*fin) $(K_1 \cap K_2) * A \in \{K_1 * A, K_2 * A, (K_1 * A) \cap (K_2 * A)\}$.

(K*fin) forces a constraint between the revision of a theory and the revision of theories in which it is included. We can indeed show that in a finite language, (K*1)-(K*8) and (K*fin) completely characterize analytic revision functions.

Theorem 5.1 Given a finite propositional language L, an operator $*$ satisfies postulates (K*1)-(K*8) and (K*fin) if and only if there exists an analytic revision function $\bar{\bullet}$ such that for any $K \in I\!\!K$, $A \in L$, $K * A = K \bar{\bullet} A$.

Proof. By Theorem 4.6 we know that $\bar{\bullet}$ validates (K*1)-(K*8). We shall verify that $\bar{\bullet}$ also validates (K*fin).

Let M be any model for $\bar{\bullet}$ $M = \langle W, d\rangle$, A any formula of L and K any theory of L such that $K = K_1 \cap K_2$ for theories K_1, K_2.

We have to show that in model M, $[K \bar{\bullet} A] \in \{[K_1 \bar{\bullet} A], [K_2 \bar{\bullet} A], [(K_1 \bar{\bullet} A)] \cup [(K_2 \bar{\bullet} A)]\}$.

By Definition 3.3 $[K \bar{\bullet} A] = \{v \in [A] : d([K], v) = d([K], [A])\}$. Also by definition, $d([K], v) = d([K_1] \cup [K_2], v) = \min\{d([K_1], v), d([K_2], v)\}$ and $d([K], [A]) = d([K_1] \cup [K_2], [A]) = \min\{d([K1], [A]), d([K_2], [A])\}$.

Then either $d([K_1], [A]) < d([K_2], [A])$ and $K \bar{\bullet} A = K_1 \bar{\bullet} A$, or $d([K_2], [A]) < d([K_1], [A])$ and $K \bar{\bullet} A = K_2 \bar{\bullet} A$, or $d([K_1], [A]) = d([K_2], [A])$ and then $K \bar{\bullet} A = K_1 \bar{\bullet} A \cap K_2 \bar{\bullet} A$.

By Theorem 4.8, given a fixed theory K, $*$ restricted to K is a analytic revision function, but a priori, with respect to different models M_K, one for each theory. We want to prove that this family of functions can actually be obtained from a single update model. i.e. that when considered as a binary function, $*$ can be obtained in the semantic framework of analytic revision.

Take the following model, $M = \langle W, d\rangle$ where W is the set of complete, consistent theories of the language and d is defined as $d(w, v) = d_w(v)$, for d_w a function characterizing the behaviour of $*$ when taking w fixed as first parameter. Also $d(\emptyset, v) = d_\emptyset(v)$. We extend d to a function on sets as we did before, by means of the min function. We now proceed by induction on the size of K.

Clearly, if K is empty or a singleton, $K * A = K \bar{\bullet} A$, by definition of d. Suppose K is not a singleton.

$[K\bar{\bullet}A \subseteq K * A]$. We want to show that if $w \in [K * A]$ then $w \in [K] \bullet [A]$. Clearly, $K\bar{\bullet}A = K * A$ for $[K]$ a singleton or $[K]$ the empty set.

Assume $[K] = \{x_1, \ldots, x_n\}$, $v \in [K*A]$ and $v \notin [K\bar{\bullet}A]$. Since $*$ validates (K*fin) and K is finite, then there must be some x in $[K]$ such that $v \in [x*A]$. Let IN $= \{x \in [K] : v \in [x * A]\}$. Also, by Definition 3.3 there must exist some $y \in [K]$ such that $d(y, [A]) = d([K], [A])$. Then $v \notin \{y\} \bullet [A]$. Hence $v \notin [y * A]$. Let OUT $= \{y \in [K] : v \notin [y * A]\}$.

Consider the following sets of two elements, $\{y_1, y_2\} \subseteq$ OUT, then trivially, by an application of (K*fin) $v \notin \{y_1, y_2\} * A$. Take now $\{x, y\}$ such that $x \in$ IN and $y \in$ OUT, then either (1) $d(x, [A]) < d(y, [A])$ or (2) $d(x, [A]) = d(y, [A])$ or (3) $d(x, [A]) > d(y, [A])$. But (1) is impossible since $x, y \in [K]$ and $d(y, [A]) = d([K], [A])$. If (2) holds then, (using that $v \in [x * A]$), $d(y, [A]) = d(x, v)$. Therefore, $v \in [K] \bullet [A]$, contrary to our assumption. Then (3) should be the case for any pair x, y. According to our definition of d, $c^{\{y\}}(A) = c^{\{x,y\}}(A)$ and $c^{\{x\}}(A) \neq c^{\{x,y\}}(A)$. Hence $\{x, y\} * A = y * A$, therefore, $v \notin \{x, y\} * A$.

Now we are almost done. Notice that by pairing elements of IN with elements of OUT we can "delete" the elements of IN from $[K]$. I.e. let $x \in$ IN, $y \in$ OUT and write $[K]$ as $\{x, y\} \cup ([K] \setminus \{x\})$, then applying (K*fin) $v \in [\mathrm{Th}([K] \setminus \{x\}) * A]$. Because IN is finite, we will finally have $v \in [\mathrm{Th}(\mathrm{OUT}) * A]$. A contradiction.

$[K * A \subseteq K\bar{\bullet}A]$. Let $u \in [K\bar{\bullet}A]$ and let $x \in [K]$ such that $d([K], [A]) = d(x, u)$. Then $u \in [x * A]$. Also, because K is finite, by repeatedly applying (K*fin) we have $[K * A] = \bigcup [T_i * A]$ for some complete theories T_i extending K. If $x = T_i$ for some i we are done. Suppose $u \notin [T_i * A]$ for any T_i. We now use again (K*fin) and comparison of pairs to arrive to a contradiction (write $[K] = \{x, T_i\} \cup ([K] \setminus \{T_i\})$ and consider $K * A \subseteq \mathrm{Th}(\{x, T_i\}) * A$ must hold for each T_i). Full details are given for the case of infinite languages in Theorem 5.3. QED

The general case is slightly harder. Postulates (K*1)-(K*8) and (K*fin) do not fully characterize the $\bar{\bullet}$ operation in a language with an infinite number of propositional letters.

Observation 5.2 Consider an infinite propositional language L. Postulates (K*1)-(K*8) and (K*fin) do not fully characterize the $\bar{\bullet}$ operation.

Proof. Given a propositional language L with an infinite but countable number of propositional letters we will exhibit a function $*$ satisfying postulates (K*1)-(K*8) and (K*fin) for which there is no model $M = \langle W, d \rangle$, satisfying that $\forall K \in \mathbb{K}$, $\forall A \in L, K * A = K\bar{\bullet}A$. We define $*$ semantically as follows. Let $K \in \mathbb{K}$, $A \in L$ and $v \in [A]$, then

$$[K * A] = \begin{cases} [K] \cap [A] & , \text{ if } [K] \cap [A] \neq \emptyset. \\ \{v\} & , \text{ if } [K] \cap [A] = \emptyset \text{ and } [K] \text{ is finite.} \\ [A] & , \text{ if } [K] \cap [A] = \emptyset \text{ and } [K] \text{ is infinite.} \end{cases}$$

For each incomplete theory $K \in I\!\!K$ such that $[K]$ has a finite number of elements (i.e., there are only a finite number of maximal consistent sets extending K), then let $*_K$ be a *fixed* AGM maxichoice revision function for K always returning one and the same maximal consistent set of A. And for each incomplete theory $K \in I\!\!K$ such that $[K]$ has an infinite number of elements then let $*_K$ be the full meet revision function for K, namely $K * A = \mathrm{Cn}(A)$.

Clearly $*$ validates (K*fin). If $[K]$ is finite it is easily verified. If $[K]$ is infinite, for any theories K_1, K_2 such that $K = K_1 \cap K_2$, either $[K_1]$ or $[K_2]$ are infinite. Then either $K_1 * A = \mathrm{Cn}(A)$ or $K_2 * A = \mathrm{Cn}(A)$, as required.

Suppose for contradiction that there is a model $M = \langle W, d\rangle$ such that for every $K \in I\!\!K$, for every $A \in L$, $K * A = \mathrm{Th}(\{y \in [A] : d([K], A) = d([K], y)\})$.

According to our definition of $*$, for every theory K such that $[K]$ is finite, if $[K] \cap [A] = \emptyset$ then $[K * A] = \{v\}$. Therefore d must verify that $\forall x \in W, d(x, x) = 0$; $\forall x, w \in W, w \neq v$, $d(x, v) < d(x, w)$.

For any $[K]$ such that $[K] \cap [A] = \emptyset$, Then $0 < d([K], [A]) = d([K], v)$, since for each $x \in [K]$, $d(x, v) = d(x, [A])$. Then $[K * A] = \{y \in [A] : d([K], A) = d([K], y)\}= \{v\}$. This contradicts the case when $[K]$ is infinite, because according to our definition $[K * A] = [A]$. QED

(K*fin) gives us the following insight: when performing the analytic revision of K by A, we should hear the opinions of the theories to which K can be extended. If we now turn to the way $\bullet$ is defined given K and A, we see that we can always identify an element w of $[K]$ which is responsible for defining $d([K], [A])$. Then $[K] \bullet [A]$ is obtained as the subset of $[A]$ standing at the same distance from $[K]$ as w is. These complete theories are clearly the ones we should pay attention to. Following this intuition we propose:

(K$*\exists$) $K * A = \bigcap (T_i * A)$, for some complete theories T_i extending K.

(K$*\forall$) If $K \subseteq K' \subseteq T$, for T a complete theory then, for all A, $K * A \subseteq T * A$ implies $K * A \subseteq K' * A \subseteq T * A$.

(K$*\exists$) claims there are some complete theories — "the intended interpretations" of our theory — that determine the result of the revision. (K$*\forall$) expresses the primacy of these complete theories and establishes a restricted form of monotony for the $*$ operator. In particular, if our theory K is regarded as an intersection of two larger theories K_1 and K_2, then (K$*\exists$) and (K$*\forall$) constrain the revision of K in terms of the other two. By (K$*\exists$) the revision of each K is guided by some complete theories. These complete theories either extend K_1 or K_2 or both. Then, by (K$*\forall$) the revision of K is included in the revision of K_1 or in the revision of K_2, or both. Notice that, in the presence of (K*1)-(K*8), the postulates (K$*\exists$) and (K$*\forall$) imply (K*fin).

We now prove that the eight AGM postulates plus (K$*\exists$) and (K$*\forall$) completely characterize the analytic revision operation. This is the most important result in this paper.

Theorem 5.3 (Representation Theorem, general case) An operator $*$ satisfies postulates (K*1)-(K*8), (K$*\exists$) and (K$*\forall$) if and only if there exists a model $M = \langle W, d\rangle$, where d is a distance function and for any $K \in I\!K$, $A \in L$ $K * A = K \bar{\bullet} A$.

Proof. We have proved in Theorem 4.6 that $\bar{\bullet}$ satisfies postulates (K*1)-(K*8). That $\bar{\bullet}$ validates (K$*\exists$) follows immediately from Definition 3.2, since min requires the existence of elements in $[K]$ such that their distance to $[A]$ is minimal. $\bar{\bullet}$ also validates (K$*\forall$) since for any Y if $x \in [K]$ and $d([K], Y) = d(x, Y)$ then $d(x, Y) = \min_{z\in[K]}\{d(z, Y)\}$. Therefore, for all $X \subseteq [K]$, if $x \in X$ then $d(X, Y) = d(x, Y)$ and $d(X, Y) = d([K], Y)$ as required. This proves the right to left implication.

Let's see the left to right part. Let $*$ be a change function satisfying (K*1)-(K*8),(K$*\exists$) and (K$*\forall$). We will construct a analytic revision model $M = \langle W, d\rangle$ which corresponds to $\bullet$.

We have to show that $\forall K \in I\!K, \forall A \in L, K * A = K\bar{\bullet}A$. We start by defining the model M. The domain W will be the set of all complete theories in the language L. To define the distance function d, let $\{S^K\}$ be the family of systems of spheres corresponding to $*$. If S^K is a given system of sphere we note as S^K_i a particular element of it, and for a given formula A, $c^K(A)$ is the minimal sphere in S^K with nonempty intersection with $[A]$.

As before, we start by determining the value of d for elements in W and then extend the function to subsets of W as in Definition 3.2. Any function $d : \mathcal{P}(W) \times \mathcal{P}(W) \to I\!R^+$ satisfying the following restrictions is appropriate.

i. $\forall v \in W$, $d(v, v) = 0$.
ii. $\forall v, u, m, d(v, u) < d(v, m)$ iff $\exists S^v_1, S^v_2 \in S^v u \in S^v_1, m \in S^v_2 \& S^v_1 \subset S^v_2$.
iii. $\forall v, u, m, d(v, u) = d(v, m)$ iff $\forall S^v_i \in S^v u \in S^v_i \Leftrightarrow m \in S^v_i$.
iv. $d(\{x, y\}, X) = d(x, X)$ iff $c^{\{x\}}(X) = c^{\{x,y\}}$.
v. $d(x, X) < d(y, X)$ iff $c^{\{x\}}(X) = c^{\{x,y\}}(X)$ and $c^{\{y\}}(X) \neq c^{\{x,y\}}(X)$.
vi. $d(x, X) = d(y, X)$ iff $c^{\{x\}}(X) \cup c^{\{y\}}(X) = c^{\{x,y\}}(X)$.
vii. $\forall v, u, m, d(\emptyset, u) < d(\emptyset, m)$ iff $\exists S^\emptyset_1, S^\emptyset_2 \in S^\emptyset u \in S^\emptyset_1, m \in S^\emptyset_2 \& S^\emptyset_1 \subset S^\emptyset_2$.

It is clear that by case vii), when K is the inconsistent theory $K * A$ and $K\bar{\bullet}A$ agree. Furthermore if $[A] = \emptyset$, by $(K * 5)$, $K * A = L$, and also $K\bar{\bullet}A = L$ by definition. We will now prove, for K and A consistent, that $u \in [K * A]$ iff $u \in [K\bar{\bullet}A]$ by analyzing the different cases.

Suppose $[K] = \{v\}$.

$[K\bar{\bullet}A \subseteq K * A]$. Let $u \in [K * A]$, to prove (1) $u \in \{w \in [A] : d(w, [A]) = d(v, w)\}$. Let $m \in [A]$ be such that $d(v, [A]) = d(v, m)$, then (1) is equivalent to (2) $d(v, m) = d(v, u)$. By iii) we have to prove that for all $S_i^{\{v\}} \in S^{\{v\}}, u \in S_i^{\{v\}} \Leftrightarrow m \in S_i^{\{v\}}$. As $d(v, [A]) = d(v, m)$ then $m \in c^{\{v\}}(A)$. Let $S_i^{\{v\}}$ be any. If $c^{\{v\}}(A) \subseteq S_i^{\{v\}}$ then both m and u are in $S_i^{\{v\}}$. If $S_i^{\{v\}} \subset c^{\{v\}}(A)$, then $u \notin S_i^{\{v\}}$. Suppose $m \in S_i^{\{v\}}$, but then $d(v, [A]) > d(v, m)$, a contradiction.

$[K * A \subseteq K\bar{\bullet}A]$. To prove the other inclusion, let $u \in [A]$ and suppose $d(v, m) = d(v, u)$ for $m \in [A]$ such that $d(v, [A]) = d(v, m)$. Suppose $u \notin c^{\{v\}}(A)$. Then by iii) $m \notin c^{\{v\}}(A)$. Let $S_i^{\{v\}}$ be the $\subseteq$-smallest such that $m \in S_i^{\{v\}}$, $c^{\{v\}}(A) \subset S_i^{\{v\}}$. By the limit assumption $c^{\{v\}}(A)$ is defined and let $m' \in c^{\{v\}}(A) \cap [A]$. But then by i) $d(v, m') < d(v, m)$ contradicting the selection of m.

The general case, $[K] > 1$.

$[K * A \subseteq K\bar{\bullet}A]$. Let $u \in [K\bar{\bullet}A]$ and let $x \in [K]$ be such that $d([K], [A]) = d(x, u)$ (notice that then, $u \in [x\bar{\bullet}A]$ and by the previous case $u \in [x * A]$). By (K$*\exists$), $[K * A] = \bigcup[T_i * A]$ for some complete theories extending K. If for some i, $u \in [T_i * A]$ we are done, so assume $u \notin [T_i * A]$ for all i.

Consider for any i the proposition $\{x, T_i\} \subseteq [K]$. Then by (K$*\forall$), $[T_i * A] \subseteq [\mathrm{Th}(\{x, T_i\}) * A] \subseteq [K * A]$. Apply (K$*\exists$) to $\mathrm{Th}(\{x, T_i\}) * A$ now. If $[x * A] \subseteq [\mathrm{Th}(\{x, T_i\}) * A]$ we are done. Rests to consider the case when $[\mathrm{Th}(\{x, T_i\}) * A] = [T_i * A]$, and furthermore $[\mathrm{Th}(\{x, T_i\}) * A] \neq [x * A]$. But then by condition v), $d(T_i, [A]) < d(x, [A])$, contradicting the choice of x.

$[K\bar{\bullet}A \subseteq K * A]$. For this inclusion, we should further prove the case for $[K] = \{v, w\}$ separately. Suppose $u \in [K * A]$, then by (K$*\exists$), $u \in \bigcup[T_i * A]$ for some T_i complete theories extending K, either

a. $K * A = v * A$. Then by iii), $d(\{v, w\}, [A]) = d(v, [A])$. As $u \in c^{\{v,w\}}(A)$, by definition of d, i) and ii) we have that $d(v, u) = d(v, [A]) = d([K], [A])$. Hence $u \in [K] \bullet [A]$.

b. $K * A = w * A$. Similar to a.

c. $K * A = v * A \cap w * A$. By iv), $d(v, [A]) = d(w, [A])$. Also, either $u \in c^{\{v\}}(A)$ or $u \in c^{\{w\}}(A)$. Hence, as above, either $d(v, u) = d(v, [A])$ or $d(w, u) = d(v, [A])$. In both cases, $u \in [K] \bullet [A]$.

$[K] > 2$. Suppose $u \in [K * A]$, then by (K$*\exists$), $u \in \bigcup[T_i * A]$ for some T_i complete theories extending K. In particular, let $T_i \in W$ be such that $u \in [T_i * A]$.

Let x be any in $[K]$, by (K$*\forall$), $K * A \subseteq T_i * A$ implies $(T_i \cap x) * A \subseteq T_i * A$.

Hence, $(T_i \cap x) * A \subseteq T_i * A$. We are now in the previous cases, of revising theories whose proposition has cardinality one or two. Therefore we can claim that $(T_i \cap x)\bar{\bullet}A \subseteq T_i\bar{\bullet}A$. I.e., by definition for all $w \in [A]$, $d(\{T_i, x\}, [A]) = d(\{T_i, x\}, w)$ then $d(T_i, [A]) = d(\{T_i, x\}, w)$, iff for all $w \in [A]$, $\min\{d(T_i, [A]), d(x, [A])\} = \min\{d(T_i, w), d(x, w)\}$ then $d(T_i, [A]) = d(T_i, w)$.

Therefore $d(T_i, [A]) = d(\{T_i, x\}, [A])$. As this is true for all $x \in [K]$, $d(T_i, [A]) = d([K], [A])$. Because $u \in [T_i \bar{\bullet} A]$, $d(T_i, u) = d(T_i, [A])$ and $u \in K \bar{\bullet} A$. QED

Theorems 5.1 and 5.3 are interesting because they give general characterization results for AGM revisions based on pseudo-distances, for both, the finite and the general cases.

We now turn our attention to two natural constraints on the distance functions which give rise to proper subclasses of analytic AGM revisions. One is to consider a distance function $d : W \times W \to I\!R^+$ is such that no two points are at the same distance from a given point, if $d(v, u) = d(v, w)$ then $v = w$. This is to take d_v, the the projection of the distance function over its first argument, to be injective. It is quite strightforward to prove that such a distance function gives rise to an analytic AGM revision that takes consistent complete theories to consistent complete theories. For complete theories this analytic function behaves as a maxichoice AGM revision. For this reason we name it *maxi-analytic AGM functions*, and we show that they are characterized by the following postulate.

(K*M) If K is consistent and complete then, for any A, $K * A$ is complete.

Observation 5.4 (maxi-analytic AGM functions) An operator $*$ satisfies postulates (K*1)-(K*8), (K*∃) (K*∀) and (K*M) if and only if there exists a distance model $M = \langle W, d \rangle$, such that for each $v \in W$, $d_v = d(v, w)$ is injective, and for any $K \in I\!K$, $A \in L$ $K * A = K \bar{\bullet} A$.

Proof. The characterization result follows directly for the fact that for every nameable $Y \subseteq W$, $\{x | d_v(Y) = d_v(x)\}$ is a singleton. QED

Another interesting consideration is the case of well founded distances, that is distances that are definable over the ordinals, $d : W \times W \to \mathcal{O}$. Applying Observation 2.6, a well founded system of spheres centered in $[K]$ can be represented by ordinal function $d_K : W \to \mathcal{O}$. In this setting actual values of the function $d(w, v)$ can be obtained by counting the number of ancestors of the argument along the well founded system of spheres centered in $\{w\}$. The class of AGM revision functions definable over well founded system of spheres has been characterized by [15]. They are called *well behaved revision functions* and they are characterized by postulates (K*1)-(K*8) plus

(K*WB) For every nonempty set X of consistent formulae of L there exists a formula $A \in X$ such that $\neg A \notin K * (A \vee B)$, for every $B \in X$.

Well behaved analytic AGM functions satisfy (K*1)-(K*8),(K∗∃),(K∗∀) and (K*WB), and are a proper subclass of general analytic functions that can be characterized semantically by a distance function d over the ordinals.

Of course, this characterization carries over analytic functions and update functions.

It is apparent from the proofs of Theorems 5.1 and 5.3 that the distance function that we use is just a convenient means to express the comparative relations relative to sets, that are induced from the comparative relations relative to single points. In fact the analytic operation can be regarded as a particular case of a more general framework. Consider a model with two ordering relations, $\langle W, \{\preceq^1_w : w \in W\}, \{\preceq^2_X : X \in \mathcal{P}(W)\}\rangle$, being $\preceq^1$, $\preceq^2$ possibly independent (total) preorders on W. Then the $\bullet$ operation would be a double minimization over the two relations, defined as

$$\min_{\preceq^2_X} \bigcup_{x \in X} \min_{\preceq^1_x}(Y)$$

where $\min_{\preceq}(V) = \{v \in V : \forall z \in V, v \preceq z\}$. Our definition of analytic revision in terms of distances obtains in this general framework, by considering $\preceq^1$ as an ordering encoding $d : W \times W \to I\!\!R^+$ and $\preceq^2$ as one encoding the extension $d : \mathcal{P}(W) \times \mathcal{P}(W) \to I\!\!R^+$. We believe it is interesting to study characterization results for the double minimization operation on the general framework. This seems to be the proper setup to investigate which are the needed properties connecting the two orderings as well as the particular properties of each of them.

6 Properties

Theorem 5.3 makes clear that analytic revision functions are a proper subset of AGM functions, those that enforce a special relation between the revision of different theories. Interestingly, inside the AGM framework we find two nice examples of functions enforcing a strong dependency on the revison of different theories. One is the *AGM full meet revision* [2, 1]. It was considered a limiting case of acceptable revision functions because it returns theories that are too small, and it is almost a constant function when the second argument is held fixed:

(Full Meet) For every $K \in I\!\!K$ such that $\neg A \in K$, $K * A = \mathrm{Cn}(A)$.

The other example arises from the work of Alchourrón and Makinson in [3] where they define the *safe contraction* function and state properties of the intersection and union of theories. A generalization of those properties is provided by the so called postulate (K*9) which counts as simple way of linking the revisions of all different theories, [4, 17]. Our analytic function imposes a more subtle dependence on the revision different theories than that of (K*9).

Observation 6.1 The following properties are not validated by the analytic revision operation.

(Weak Intersection) If $\neg A \in K_1 \cap K_2$ then $(K_1 \cap K_2) * A = (K_1 * A) \cap (K_2 * A)$.
(Union) $(K_1 \cup K_2) * A = (K_1 * A) \cup (K_2 * A)$.
(Weak Union) If $\neg A \in K_1 \cap K_2$ then $(K_1 \cup K_2) * A = (K_1 * A) \cup (K_2 * A)$.
(K*9) If $\neg A \in K,\ K * A = L * A$.

PROOF. We prove Weak Intersection. Let a propositional language L with just two letters A and B. Let $[A] = \{w_1, w_2\}$, $[B] = \{w_2, w_3\}$, and $[K_1] = \{w_3\}$ and $[K_2] = \{w_4\}$. Let $d(w_i, w_i) = 0$, $d(w_i, w_j) = i$ if i is odd $d(w_i, w_j) = j$ if i is even. Thus, $[K_1] \bullet [A] = \{w_1, w_2\}$, $[K_2] \bullet [A] = \{w_1\}$, so $\{w_1, w_2\} = [K_1] \bullet [A] \cup [K_2] \bullet [A] = [(K_1 \bullet A) \cap (K_2 \bullet A)]$. And $[K_1 \cap K_2] \bullet [A] = \{w_3, w_4\} \bullet [A] = \{w_1\}$. Therefore $(K_1 \cap K_2) \bullet A \neq (K_1 \bullet A) \cap (K_2 \bullet A)$. QED

Let's turn now to the problem of iterated application of the $*$ operation. A pertinent criticism of the AGM formalism is its lack of definition with respect to iterated change (see [10] and [19, 18]). Although the AGM formalism does not forbid the iteration of change functions, it omits any specification of how it should be performed or what the properties of successive change are. AGM functions $* : I\!\!K \times L \to I\!\!K$ can be iterated, that is, $(K * A) * B$ is well defined. But since there are no properties linking the revision of different theories the result can be erratic. Our analytic revisions do impose some dependence between the revision of different theories, for what they can be candidate functions for iterated theory revision. Analytic revisions inherit the form of iteration of the standard update operation. The formal structure $M = \langle W, d \rangle$ determines the distance from every $[K]$. Since the analytic revision of K by A is a theory $K \bar{\bullet} A$, also a proposition in the same model M, distance from $K \bar{\bullet} A$ is also defined in the structure. We can prove that analytic revisions satisfy some natural conditions of iterated change.

(Or-Left) If $D \in (K * (A \vee B)) * C$ then $D \in (K * A) * C$ or $D \in (K * B) * C$.

(Or-Right) If $D \in (K * A) * C$ and $D \in (K * B) * C$ then $D \in (K * (A \vee B)) * C$.
And in general they fail

(Commutativity) $(K * A) * B = (K * B) * A$.

Observation 6.2 Analytic revision functions satisfy Or-Left and Or-Right and fail Commutativity.

PROOF. Let's name $X = [K] \bullet [A]$, $Y = [K] \bullet [B]$.
(Or-Left). $[K] \bullet [A \vee B] \bullet [C] = \{w \in [C] : \min_{\{x \in [K] \bullet [A \vee B]\}} \min_{\{y \in [C]\}} \{d(x, y)\}\} =$ (by (K*7) and (K*8)) $[K] \bullet [A \vee B] = [K] \bullet [A]$, or $[K] \bullet [A \vee B] = [K] \bullet [B]$, or $[K] \bullet [A \vee B] = ([K] \bullet [A]) \cup ([K] \bullet [B])$. Then, either
(1) $\{w \in [C] : \min_{\{x \in X\}} \min_{\{y \in [C]\}} \{d(x, y)\}\} = [K] \bullet [A]$; or
(2) $\{w \in [C] : \min_{\{x \in Y\}} \min_{\{y \in [C]\}} \{d(x, y)\}\} = [K] \bullet [B]$; or
(3) $\{w \in [C] : \min_{\{x \in X \cup Y\}} \min_{\{y \in [C]\}} \{d(x, y)\}\} =$

$\{w \in [C] : \min\{\min_{\{x \in X\}} \min_{\{y \in [C]\}}\{d(x,y)\}, \min_{\{x \in Y\}} \min_{\{y \in [C]\}}\{d(x,y)\}\}\}$
is either equal to $[K] \bullet [A]$ or it is equal to $[K] \bullet [B]$.

(Or-Right). Assume (1) $D \in (K \bar{\bullet} A) \bar{\bullet} C$ and (2) $D \in (K \bar{\bullet} B) \bar{\bullet} C$.
By (1) $\{w \in [C] : \min_{\{x \in X\}} \min_{\{z \in [C]\}}\{d(x,z)\}\} \subseteq [D]$.
By (2) $\{w \in [C] : \min_{\{y \in Y\}} \min_{\{z \in [C]\}}\{d(y,z)\}\} \subseteq [D]$.
And $[K] \bullet [A \vee B] \bullet [C] =$
$\{w \in [C] : \min_{\{x \in X \cup Y\}} \min_{\{z \in [C]\}}\{d(x,z)\}\} = \{w \in [C] :$
$\min(\min_{\{x \in X\}} \min_{\{z \in [C]\}}\{d(x,z)\}, \min_{\{x \in Y\}} \min_{\{z \in [C]\}}\{d(x,z)\})\}$
is either equal to $[K] \bullet [A]$ or is equal to $[K] \bullet [B]$.
Then $[K] \bullet [A \vee B] \bullet [C] \subseteq [D]$.

QED

Analytic revisions provide a difinitionally simple scheme of iterated change. But, while formally attractive, they make a simplifying assumption. Each theory is modified in a predetermined way independently of how we have obtained such a theory. Analitic revisions stisfy,

(Functionality) If $H = ((K*A_1)\ldots*A_n)$ then $H*A = ((K*A_1)\ldots*A_n)*A$.

But if K is really considered an *argument* of the function $*$, this is to be expected. If f is a function, it is required that $f(a) = f(b)$ whenever $a = b$. This functional behavior has been interpreted as a *lack of historic memory*, because each theory is modified in a predetermined way independently of how we have obtained such a theory. Lehmann in [12] mentions this property as a "Non Postulate," for he considers that interesting iterated systems should not make this strong simplifying assumption. Proposals that deal with iterated change and possess historic memory ought to expand the AGM model in such a way that change functions return not only the modified theory but also a modified version of the change function (or equivalently, return enough information to construct a new change function). Also in [12] Lehmann proposes seven postulates for iterated change. In our notation they are:

(I1) $K * A$ is a consistent theory.
(I2) $A \in K * A$.
(I3) If $B \in K * A$, then $A \supset B \in K$.
(I4) If $A \in K$ then $K * B_1 * \ldots * B_n = K * A * B_1 * \ldots * B_n$ for $n \geq 1$.
(I5) If $A \in \mathrm{Cn}(B)$, then $K * A * B * B_1 * \ldots * B_n = K * B * B_1 * \ldots * B_n$.
(I6) If $\neg B \notin K * A$ then $K * A * B * B_1 * \ldots * B_n = K * A * (A \wedge B) * B_1 * \ldots * B_n$.
(I7) $K * \neg B * B \subseteq \mathrm{Cn}(K \cup \{B\})$.

Condition (I7) implies dependency between two revision steps and consequently enforces (at least to some extent) the property of "historic memory" which analytic revisions lack. As remarked by Lehmann, the standard update operation fails postulates (I4), (I5) and (I7), and satisfies the rest. It is then expected that the analytic revision operation violates (I5) and (I7) and validates the rest.

Observation 6.3

i) All analytic revision functions satisfy (I1), (I2), (I3), (I4) and (I6).

ii) There exist analytic revision functions violating (I5) and (I7).

Proof. The violation of (I5) and (I7) can be proved by constructing a counterexample. (I1), (I2), (I3), (I4) follow from the AGM postulates (K*1)-(K*4). For (I6) we should prove that if $\neg B \notin K\bar{\bullet}A$ then $K\bar{\bullet}A\bar{\bullet}B = K\bar{\bullet}A\bar{\bullet}(A \wedge B)$. But this is obvious since $K\bar{\bullet}A\bar{\bullet}B = \mathrm{Cn}(K\bar{\bullet}A \cup \{B\}) = \mathrm{Cn}(K\bar{\bullet}A \cup \{A \wedge B\})$. QED

Darwiche and Pearl [8] introduce (C1)-(C4) as desirable properties of iterated revisions, while (C5) and (C6) are considered too demanding. Condition (C1) amounts to Lehmann's (I5) and condition (C2) has been proved inconsistent with the AGM postulates (K*7) and (K*8) in [12]. Analytic revisions do not validate any of Darwiche and Pearl's postulates.

(C1) If $A \in \mathrm{Cn}(B)$ then $(K * A) * B = K * B$.

(C2) If $\neg A \in \mathrm{Cn}(B)$ then $(K * A) * B = K * B$.

(C3) If $A \in K * B$ then $A \in (K * A) * B$.

(C4) If $\neg A \notin K * B$ then $\neg A \notin (K * A) * B$.

(C5) If $\neg B \in K * A$ and $A \notin K * B$ then $A \notin (K * A) * B$.

(C6) If $\neg B \in K * A$ and $\neg A \in K * B$ then $\neg A \in (K * A) * B$.

Observation 6.4 There exist analytic revision functions violating each of (C1)-(C6).

Proof. C1 is just postulate I5 above.

C2. Assume a propositional language with three variables A, B and C. Let $w \in [\neg A] \cap [\neg B]$, $z \in [A] \cap [\neg B]$, $v \in [\neg A] \cap [B] \cap [C]$ and $u \in [\neg A] \cap [B] \cap [\neg C]$. Suppose $d(z, v) < d(z, u)$ and $d(w, u) < d(w, v)$. Let $[K] = \{w\}$. Then $[K\bar{\bullet}A] = \{z\}$, $[K\bar{\bullet}A\bar{\bullet}B] = \{v\}$ but $[K\bar{\bullet}B] = \{u\}$.

C3 and C4. Let $w \in [\neg A] \cap [\neg B]$, $z \in [A] \cap [\neg B]$, $v \in [\neg A] \cap [B] \cap [C]$ and $x \in [A] \cap [B] \cap [C]$. Suppose $d(w, z) < d(w, x) < d(w, v)$ and $d(z, v) < d(z, x)$. Let $[K] = \{w\}$. Then $[K\bar{\bullet}B] = \{x\} \subseteq [A]$, $[K\bar{\bullet}A] = \{z\}$ and $[K\bar{\bullet}A\bar{\bullet}B] = \{v\} \not\subseteq [A]$.

C5 and C6. Let $w \in [\neg A] \cap [\neg B]$, $z \in [A] \cap [\neg B]$, and $x \in [A] \cap [B]$ and $u \in [\neg A] \cap [B]$. Suppose $d(w, z) < d(w, u) < d(w, x)$ and $d(z, x) < d(z, u)$. Let $[K] = \{w\}$. Then $[K\bar{\bullet}B] = \{u\} \subseteq [\neg A]$, $[K\bar{\bullet}A] = \{z\} \subseteq [\neg B]$ but $[K\bar{\bullet}A\bar{\bullet}B] = \{x\} \subseteq [A]$. QED

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