Deontic Action Logics: A Modular Algebraic Perspective

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Abstract

In a seminal work, K. Segerberg introduced deontic action logic (DAL) as a formal framework to investigate normative reasoning over actions. In this work, we revisit DAL and provide a complete algebraization for it. To this end, we introduce deontic action algebras—algebraic structures consisting of a Boolean algebra for interpreting actions, a Boolean algebra for interpreting formulas, and two mappings from one Boolean algebra to the other interpreting the deontic concepts of permission and prohibition. We show how this framework supports the derivation of various deontic action logics by imposing or relaxing structural conditions on either Boolean algebra. This flexibility allows us to uniformly account for several logics within the broader DAL family. In particular, we introduce four variations obtained by: (a) enriching the algebra of formulas with propositions on states, (b) adopting a Heyting algebra for state propositions, (c) adopting a Heyting algebra for actions, and (d) adopting Heyting algebras for both. We illustrate these new deontic action logics with examples and establish their algebraic completeness.

Keywords: Deontic Action Logic, Algebraic Logic, Normative Reasoning.

1 Introduction

The logical laws of normative or deontic reasoning have attracted the attention of philosophers, lawyers, logicians, and computer scientists since the beginning of their disciplines [1]. The first modern deontic logical systems go back to the pioneer works of von Wright [2], Kalinowski [3], and Becker [4]. These early systems were grounded in the idea that prescriptions apply to actions. However, as the influence of Modal Logic grew, a shift occurred: prescriptions increasingly came to be applied to propositions rather than actions [5]. This shift marked a pivotal moment and gave rise to two distinct traditions: on one side we have *ought-to-be* systems, where normative expressions apply to propositions; on the other, we have *ought-to-do* systems, which retain the original intuition that norms are fundamentally about actions.

The most prominent system in the *ought-to-be* tradition is *Standard Deontic Logic* (SDL), i.e., the basic normal modal system K extended with the axiom for *seriality* [6, 7]. In SDL, the modal "diamond" is written $\mathsf{P}\varphi$ and is informally interpreted as capturing "φ is permitted". Using this modality, we can formally analyze how to deal with notions of permission and prohibition. For example, let w and i express the propositions "John is behind the steering wheel" and "John is intoxicated". Then, we can ask which of the following best captures the intended meaning of "it is not permitted that John is intoxicated behind the steering wheel": $w \to \neg\mathsf{P}(i)$, $\neg\mathsf{P}(w \to i)$, or $\neg\mathsf{P}(w \wedge i)$. This kind of question have been extensively explored in the literature on Deontic Logic, and remain a source of ongoing debate, riddled with challenges and paradoxes [6]. The SDL framework has been generalized to a variety of modal logics designed to capture increasingly complex normative scenarios. In this line, we find PDL-based formalisms like those in [29**?**], which model prescriptions over actions by evaluating whether executing a given action results in an undesirable state of affairs, such as incurring a sanction. Along similar lines, we find the highly expressive family of STIT logics (see, e.g., [**? ? ? ?**]), which provide tools to describe notions such as modes of responsibility in action execution, as well as concepts like commitment and the achievement of obligations.

The *ought-to-do* counterpart to SDL, and its relatives, is arguably the *Deontic Action Logic* (DAL) proposed by Segerberg in [8]. This deontic logic takes a different starting point: rather than adding to the modal machinery, it shifts the focus to the structure of actions themselves. Precisely, the language of DAL draws a clear distinction between actions and formulas. Actions are constructed from basic action names, such as $\mathsf{drinking}$ and $\mathsf{driving}$, which represent the acts of drinking and driving, respectively. These can be combined using action operators, such as $\mathsf{drinking} \sqcap \mathsf{driving}$, which denotes the compound action of drinking while driving. In DAL's setting, the deontic predicates P (for permission) and F (for forbidden/prohibition) are then applied to actions to form formulas, which can in turn be combined using standard logical connectives. For instance, the formula $\neg\mathsf{P}(\mathsf{driving} \sqcap \mathsf{drinking})$ expresses the proposition that drinking while driving is not permitted. In this way, formulas of DAL align closely with the intuition that norms govern what agents do rather than what merely holds.

The work of Segerberg in [8] sets the stage for a broader family of deontic action logics [8–15]. These logics have proven useful not only from a theoretical standpoint, but also as frameworks for modeling and reasoning about the behavior of real-world systems. For instance, in [11, 16], a variant of DAL is used to reason about fault-tolerance. Therein, actions formalize changes of state in a system, permitted actions indicate the normal behavior of the system, while forbidden actions are used to model the faulty behavior of the system. This classification of actions paves the way to reasoning about how to react in response to faults. In this setting, we can, for instance, understand a formula such as $\mathsf{F}(\mathsf{read} \sqcap \mathsf{write})$ as prescribing the behavior of a system by indicating that it is forbidden to simultaneously read and write from a memory location, as this could lead to the system being in an inconsistent state. The falsehood of this formula in a particular scenario signals faulty behavior, indicating the necessity of fault-tolerant mechanisms to safeguard the normal operation of the system.

Proposal. We take DAL as the starting point to investigate the construction of deontic action logics in an algebraic framework. To this end, we build on an earlier work where we develop an abstract view of DAL resorting to algebraic structures called *deontic action algebras* [17]. In brief, a deontic action algebra consists of a Boolean algebra for interpreting actions, a Boolean algebra for interpreting formulas, and two mappings from one Boolean algebra to the other interpreting the deontic concepts of permission and prohibition. We use deontic action algebras to formally interpret the two tier structure of the language of DAL. An interesting feature of this algebraic treatment of DAL is that it is modular, giving rise to natural variations. We explore this modularity by elaborating of how to capture various logics in the DAL family.

Contributions. This paper continues and extends our work in [17]. First, we revisit the algebraic framework of deontic action algebras, provide detailed soundness and completeness proofs, and add motivating examples. Second, we present a series of deontic action logics which exploit the modularity of the algebraic framework. We begin by enriching the algebra for formulas with propositions to describe *states of affairs*. In this way, we can express both properties of actions, and propositions (e.g., pre- and post-conditions) about the states in which this actions take place. Then, we discuss the result of replacing the Boolean algebra interpreting formulas by a Heyting algebra. The resulting extension of DAL can deal with scenarios in which laws like the excluded middle, contraposition, or elimination of double negation are possibly rejected. This could be the case, for example, in normative systems in which evidence is required in order to accept some assertions as true. In turn, we consider using a Heyting algebra to interpret actions. We argue that this would be useful, e.g., when actions are associated to constructions witnessing their *realizability*. This admits a direct analogy with the standard interpretation of intuitionistic logic in which the concept of truth is associated to that of proof. Finally, we present a fully intuitionistic deontic action logic were both actions and formulas are interpreted using Heyting algebras. In all cases, we obtain axiom systems that are sound and complete for the corresponding classes of deontic action algebras. These results illustrate how the algebraic approach provides a uniform and flexible foundation for extending DAL, yielding a series of well-motivated and complete logics within a coherent semantic framework.

Structure of the Paper. In Sec. 2 we introduce Segerberg's deontic action logic DAL. In Sec. 3 we present the basic algebraic framework of deontic action algebras, and prove soundness and completeness for DAL using standard algebraic tools. This section primarily reviews and systematizes existing results. In Sec. 4, we investigate variations of DAL within the algebraic framework of deontic action algebras. Finally, in Sec. 5, we offer some final remarks and discuss future work.

2 Deontic Action Logic

In this section, we revise the basics of Segerberg's Deontic Action Logic (DAL) [8]. Precisely, we introduce the language of actions and formulas, their interpretation via deontic action models, and a Hilbert-style axiomatization of the logic. To support later developments, we also present a soundness and completeness result that will serve as a reference point throughout the paper.

Language.

The language of DAL consists of *actions* and *formulas*. Actions, indicated α, β, γ, $\ldots$, are built on a countable set $\mathsf{Act}_0 = \{ \mathsf{a}_i \mid i \in \mathbb{N}_0 \}$ of basic action symbols according to the following grammar:

$$\alpha, \beta ::= \mathsf{a}_i \mid \alpha \sqcup \beta \mid \alpha \sqcap \beta \mid \bar{\alpha} \mid \mathsf{0} \mid \mathsf{1}. \tag{1}$$

We use Act to indicate the set of all actions. Formulas of DAL, indicated φ, ψ, χ, $\ldots$, are built on the set Act according to the following grammar:

$$\varphi, \psi ::= \alpha = \beta \mid \mathsf{P}(\alpha) \mid \mathsf{F}(\alpha) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \neg\varphi \mid \bot \mid \top. \tag{2}$$

We use Form to indicate the set of all formulas of DAL. Intuitively, an action symbol $\mathsf{a} \in \mathsf{Act}_0$ indicates a *basic* action; action $\alpha \sqcup \beta$ indicates the *free-choice* composition of α and β; action $\alpha \sqcap \beta$ indicates the *parallel* composition of α and β; and $\bar{\alpha}$ indicates complement of α. Finally, $\mathsf{0}$ and $\mathsf{1}$ indicate the *impossible* and the *universal* actions, respectively. Turning to formulas, $\alpha = \beta$ indicates that α and β are the same actions; $\mathsf{P}(\alpha)$ is read as α is *permitted*; and $\mathsf{F}(\alpha)$ is read as α is *forbidden*. Formulas built using $\wedge$, $\vee$, and $\neg$, as well as $\bot$ and $\top$, have their standard interpretation. We use $\varphi \to \psi$ as an abbreviation for $\neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

In Ex. 1, we illustrate the syntax of actions and formulas in DAL through concrete examples, and provide informal readings to clarify their intended meaning.

Example 1. *Let* $\mathsf{parking}$*,* $\mathsf{drinking}$*, and* $\mathsf{driving}$ *be basic actions in* Act_0*. Then:*

1. $\overline{\mathsf{parking}} = \mathsf{driving}$ *asserts that '(actively) driving is the complement of parking';*
2. $\mathsf{F}(\mathsf{drinking} \sqcap \mathsf{driving})$ *asserting that 'drinking while driving is forbidden';*
3. $\neg\mathsf{P}(\mathsf{drinking} \sqcap \mathsf{parking})$ *asserting that 'it is not permitted to park while drinking'.*

In the formulas above, we consider driving and parking as complementary actions. I.e., driving is the act of maintaining the car in motion (usually with the goal of reaching a desired destination), whereas parking is the act of leaving the car in a fixed position. Drinking, on the other hand, is the act of consuming alcohol. In turn, we use the formulas in items 2 and 3 as a way to introduce a subtle distinction between being forbidden and not being permitted. This distinction becomes important since DAL *do* *not assume* deontic closure—*i.e., not every action is classified as either permitted or forbidden. We will come back to this point later on.*

Semantics.

The semantics of DAL is defined over deontic action models. A deontic action model is a tuple $\mathfrak{M} = \langle E, P, F \rangle$, where E is a (possibly empty) set of elements representing realizations of actions, and $P, F \subseteq E$ are disjoint sets of permitted and forbidden realizations of actions, respectively. The disjointness condition, i.e., $P \cap F = \varnothing$, ensures that no realization of an action is both permitted and forbidden. A *valuation* on a deontic model $\mathfrak{M} = \langle E, P, F \rangle$ is a function $v : \mathsf{Act}_0 \to 2^E$, assigning to each basic action the set of its possible realizations; i.e., a valuation describes how basic actions are realized within the model.

Proposition 1. *For every deontic action model $\mathfrak{M} = \langle E, P, F \rangle$, and any valuation $v : \mathsf{Act}_0 \to 2^E$ on $\mathfrak{M}$, there is a unique $v^* : \mathsf{Act} \to 2^E$ s.t.:*

$$
\begin{array}{lll}
v^*(\alpha \sqcup \beta) = v^*(\alpha) \cup v^*(\beta) & v^*(\bar{\alpha}) = E \backslash v^*(\alpha) & v^*(0) = \varnothing \\
v^*(\alpha \sqcap \beta) = v^*(\alpha) \cap v^*(\beta) & & v^*(1) = E.
\end{array}
$$

The *satisfiability* of a formula φ on a deontic action model $\mathfrak{M} = \langle E, P, F \rangle$ under a valuation v, written $\mathfrak{M}, v \Vdash \varphi$, is defined inductively as:

$$
\begin{array}{lll}
\mathfrak{M}, v \Vdash \alpha = \beta & \text{iff} & v^*(\alpha) = v^*(\beta) \\
\mathfrak{M}, v \Vdash \mathsf{P}(\alpha) & \text{iff} & v^*(\alpha) \subseteq P \\
\mathfrak{M}, v \Vdash \mathsf{F}(\alpha) & \text{iff} & v^*(\alpha) \subseteq F \\
\mathfrak{M}, v \Vdash \varphi \vee \psi & \text{iff} & \mathfrak{M}, v \Vdash \varphi \text{ or } \mathfrak{M}, v \Vdash \psi \\
\mathfrak{M}, v \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, v \Vdash \varphi \text{ and } \mathfrak{M}, v \Vdash \psi \\
\mathfrak{M}, v \Vdash \neg\varphi & \text{iff} & \mathfrak{M}, v \nVdash \varphi \\
\mathfrak{M}, v \Vdash \bot & & \text{never} \\
\mathfrak{M}, v \Vdash \top & & \text{always.}
\end{array}
$$

We say that a formula φ is a *tautology* iff for any deontic action model $\mathfrak{M}$ and for any valuation v on $\mathfrak{M}$, it follows that $\mathfrak{M}, v \Vdash \varphi$.

In Ex. 2, we present examples of deontic action models for the formulas in Ex. 1.

Example 2. *Let E be a set with exactly three elements. Consider the deontic action model $\mathfrak{M} = \langle E, P, F \rangle$, where P and F are as defined in Fig. 1. Similarly, let $\mathfrak{M}' = \langle E, P', F \rangle$ be the model where P' and F are as in Fig. 2. Lastly, consider basic action symbols $\{\mathsf{drinking}, \mathsf{driving}, \mathsf{parking}\} \subset \mathsf{Act}_0$, and define $v : \mathsf{Act}_0 \to 2^E$ as a valuation where $v(\mathsf{drinking})$, $v(\mathsf{driving})$, and $v(\mathsf{parking})$ are as in Fig. 1. Then:*

1. $\mathfrak{M}, v \Vdash \overline{\mathsf{parking}} = \mathsf{driving}$
2. $\mathfrak{M}, v \Vdash \mathsf{F}(\mathsf{drinking} \sqcap \mathsf{driving})$
3. $\mathfrak{M}, v \Vdash \mathsf{P}(\mathsf{drinking} \sqcap \mathsf{parking})$
4. $\mathfrak{M}, v \nVdash \neg\mathsf{P}(\mathsf{drinking} \sqcap \mathsf{parking})$
5. $\mathfrak{M}', v \Vdash \overline{\mathsf{parking}} = \mathsf{driving}$
6. $\mathfrak{M}', v \Vdash \mathsf{F}(\mathsf{drinking} \sqcap \mathsf{driving})$
7. $\mathfrak{M}', v \nVdash \mathsf{P}(\mathsf{drinking} \sqcap \mathsf{parking})$
8. $\mathfrak{M}', v \Vdash \neg\mathsf{P}(\mathsf{drinking} \sqcap \mathsf{parking})$.

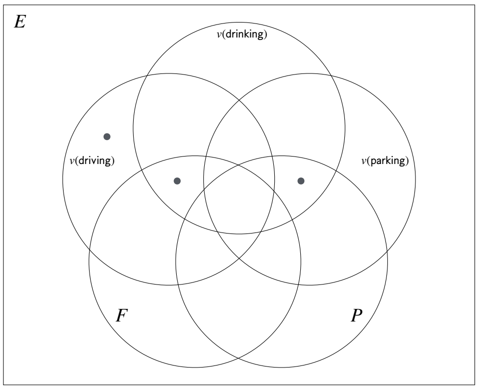

Figure 1: A Deontic Action Model.

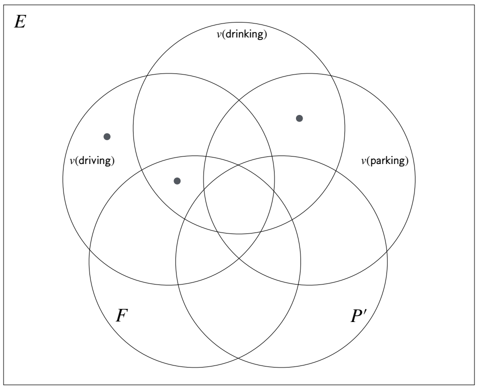

Figure 2: Anoter Deontic Action Model.

Models $\mathfrak{M}$ and $\mathfrak{M}'$ show that $\mathsf{F}(\alpha)$ *and* $\neg\mathsf{P}(\alpha)$ *are not equivalent in* DAL.

Axiomatization.

We present an axiomatization of DAL in Fig. 3. This axiomatization differs slightly from the original formulation in [8]. The modification is made for convenience, as it simplifies the presentation of the variations of DAL introduced in following sections. The axioms are organized into four groups. The first group (A1–A13 and LEM) characterizes the operations on actions and is based on the presentation of Boolean algebras as complemented distributive lattices in [18, 19]. The second group (A1'–A13' and LEM') provides analogous axioms for the propositional connectives on formulas. The third group (E1 and E2) captures equality on actions; in particular, E2 ensures that equal actions can be freely substituted for one another in a formula; in this axiom φ^{β}_{α} is obtained by replacing some occurrences of α in φ with β. Finally, the fourth group (D1–D3) defines the behavior of the deontic operators for permission P and prohibition F.

In DAL, a Hilbert-style proof of a formula φ is defined as a finite sequence $\psi_1, \ldots, \psi_n$ of formulas such that: $\psi_n = \varphi$, and for each $1 \leqslant k \leqslant n$, ψ_k is an axiom, or is obtained from two earlier formulas ψ_i and ψ_j using the rule of *modus ponens* (i.e., there are $1 \leqslant i < j < k$ such that $\psi_j = \psi_i \to \psi_k$). We say that φ is a theorem, and write $\vdash \varphi$, iff there is a proof of φ. We make a slight abuse of notation and use DAL to indicate both the logic and its set of theorems. We state Thm. 2 for future reference.

Theorem 2 ([8])**.** *In* DAL*, a formula is a theorem if and only if it is a tautology.*

3 Deontic Action Logic via Algebra

We now turn our attention to revisiting and expanding the algebraic characterization of DAL we presented in [17]. To be noted, this algebraic framework is mathematically more abstract compared to the one in [8]. This level of abstraction is a characteristic of algebraic logics, which can be leveraged to address broader issues in deontic logic.

A1. $\alpha \sqcap (\beta \sqcap \gamma) = (\alpha \sqcap \beta) \sqcap \gamma$	A8. $\alpha \sqcup (\beta \sqcup \gamma) = (\alpha \sqcup \beta) \sqcup \gamma$
A2. $\alpha \sqcap \beta = \beta \sqcap \alpha$	A9. $\alpha \sqcup \beta = \beta \sqcup \alpha$
A3. $\alpha \sqcap \alpha = \alpha$	A10. $\alpha \sqcup \alpha = \alpha$
A4. $\alpha \sqcap (\alpha \sqcup \beta) = \alpha$	A11. $\alpha \sqcup (\alpha \sqcap \beta) = \alpha$
A5. $\alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma)$	A12. $\alpha \sqcup (\beta \sqcap \gamma) = (\alpha \sqcup \beta) \sqcap (\alpha \sqcup \gamma)$
A6. $\alpha \sqcap \mathsf{0} = \mathsf{0}$	A13. $\alpha \sqcup \mathsf{1} = \mathsf{1}$
A7. $\alpha \sqcap \bar{\alpha} = \mathsf{0}$	LEM. $\alpha \sqcup \bar{\alpha} = \mathsf{1}$
A1'. $\varphi \wedge (\psi \wedge \chi) \leftrightarrow (\varphi \wedge \psi) \wedge \chi$	A8'. $\varphi \vee (\psi \vee \chi) \leftrightarrow (\varphi \vee \psi) \vee \chi$
A2'. $\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$	A9'. $\varphi \vee \psi \leftrightarrow \psi \vee \varphi$
A3'. $\varphi \wedge \varphi \leftrightarrow \varphi$	A10'. $\varphi \vee \varphi \leftrightarrow \varphi$
A4'. $\varphi \wedge (\varphi \vee \psi) \leftrightarrow \varphi$	A11'. $\varphi \vee (\varphi \wedge \psi) \leftrightarrow \varphi$
A5'. $\varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$	A12'. $\varphi \vee (\psi \wedge \chi) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
A6'. $\varphi \wedge \bot \leftrightarrow \bot$	A13'. $\varphi \vee \top \leftrightarrow \top$
A7'. $\varphi \wedge \neg\varphi \leftrightarrow \bot$	LEM'. $\varphi \vee \neg\varphi \leftrightarrow \top$
E1. $\alpha = \alpha$	E2. $(\alpha = \beta \wedge \varphi) \rightarrow \varphi^{\beta}_{\alpha}$
D1. $\mathsf{P}(\alpha \sqcup \beta) \leftrightarrow (\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta))$	D3. $(\mathsf{P}(\alpha) \wedge \mathsf{F}(\alpha)) \leftrightarrow (\alpha = \mathsf{0})$
D2. $\mathsf{F}(\alpha \sqcup \beta) \leftrightarrow (\mathsf{F}(\alpha) \wedge \mathsf{F}(\beta))$	

Figure 3: Axiom System for DAL.

Furthermore, a distinguishing feature of our approach is its modularity. The definition of the class of algebras described below can be modified to support, for example, additional deontic operators or change their intended interpretation; and standard algebraic tools can then be employed to attempt to prove soundness and completeness results. We take advantage of this feature to build new deontic actions logics in the spirit of DAL in Sec. 4.

3.1 Basic Definitions (and a Roadmap for our Results)

We provide a brief overview of some fundamental concepts in the algebraization of logic: signatures, algebras, characterizations of classes of algebras, congruences, and quotient algebras. In the case of DAL, these definitions require sorts to distinguish between actions and propositions. This led us to work with many-sorted algebras – algebraic structures with carrier sets and operations categorized into *sorts* [19, 20]. In what follows, we establish the basic terminology for many-sorted algebras in the context of the algebraization of a logic. We have two main purposes behind this: first, to introduce the notation and terminology we use and ensure our results are self-contained; second, to outline the key steps in our algebraization of DAL.

The algebraization of a logic begins with the appropriate definition of a signature and an algebraic structure.

Definition 1. *A (many-sorted) signature is a pair $\Sigma = \langle S, \Omega \rangle$ where: S is a non-empty set of sort symbols; and Ω is an S^+-indexed family of pairwise-disjoint sets of operation symbols (with S^+ the set of finite non-empty sequences over S). In turn,* *an algebra of type* Σ*, or a* Σ*-algebra, is a structure* $\mathbf{A} = \langle |\mathbf{A}|, \mathscr{F} \rangle$ *where:* $|\mathbf{A}|$ *is an* S*-indexed family of non-empty universe sets* $|\mathbf{A}|_s$*; and* $\mathscr{F}$ *is a collection of functions* $f_{\mathbf{A}} : (\prod_{i=1}^{n} |\mathbf{A}|_{s_i}) \to |\mathbf{A}|_s$*, one for each* $f \in \Omega_{s_1 \dots s_n s}$*.*

For the rest of this section, by an algebra, we mean an algebra of type $\Sigma = \langle S, \Omega \rangle$.

Signatures give rise to specific algebras whose universe sets consist of strings of symbols from the signature, and whose functions operate as concatenation of these strings. These algebras, known as *term algebras*, serve as the algebraic counterpart to the language of a logic. We introduce the precise definition of term algebra in Def. 2

Definition 2. *Let* V *be an* S*-indexed family of pair-wise disjoint countable sets of symbols for variables. The term algebra* $\mathbf{T}$ *(on* V*) is defined s.t.:*

1. *for all* $s \in S$*,* $|\mathbf{T}|_s$ *is the smallest set containing:* V_s*, and all strings '*$f(\tau_1 \dots \tau_n)$*' where* $f \in \Omega_{s_1 \dots s_n s}$*, and* $\tau_i \in |\mathbf{T}|_{s_i}$*;*
2. *for all* $f \in \Omega_{s_1 \dots s_n s}$*,* $f_{\mathbf{T}}(\tau_1 \dots \tau_n) =$ *'*$f(\tau_1 \dots \tau_n)$*'.*

The algebraic counterpart of the semantics of a logical language is given via homomorphisms of term algebras.

Definition 3. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be algebras, and* $h = \{ h_s : |\mathbf{A}|_s \to |\mathbf{B}|_s \mid s \in S \}$ *be an* S*-indexed family of functions. We say that* h *is a homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$*, and write* $h : \mathbf{A} \to \mathbf{B}$*, iff for all* $f \in \Omega_{s_1 \dots s_n s}$*,* $h_s(f_{\mathbf{A}}(a_1 \dots a_n)) = f_{\mathbf{B}}(h_{s_1}(a_1) \dots h_{s_n}(a_n))$*. An interpretation of a term algebra* $\mathbf{T}$ *with variables in* V *on an algebra* $\mathbf{A}$ *is a homomorphism* $h : \mathbf{T} \to \mathbf{A}$*.*

To establish soundness and completeness results in an algebraic way we will need to connect term algebras to particular classes of algebras of interest. Standard classes of algebraic structures are characterized by equations. However, our algebraization of DAL uses the weaker notion of a quasi-equation to capture equality on actions as an algebraic operation. We introduce these concepts next.

Definition 4. *Let* $\mathbf{T}$ *be a term algebra with variables in* V*. An equation is a string* $\tau_1 \doteq_s \tau_2$ *where* $\tau_i \in |\mathbf{T}|_s$*. In turn, a quasi-equation is a string* $\tau_1 \doteq_s \tau_2 \Rightarrow \tau_1' \doteq_{s'} \tau_2'$ *where* $\tau_i \in |\mathbf{T}|_s$ *and* $\tau_i' \in |\mathbf{T}|_{s'}$*. By* $\tau_1 \doteq_s \tau_2 \Leftrightarrow \tau_1' \doteq_{s'} \tau_2'$*, we mean the pair of quasi-equations* $\tau_1 \doteq_s \tau_2 \Rightarrow \tau_1' \doteq_{s'} \tau_2'$ *and* $\tau_1' \doteq_{s'} \tau_2' \Rightarrow \tau_1 \doteq_s \tau_2$*.*

In particular, notice that an equation of the form $\tau_1 \doteq_s \tau_2$ *can be stated as the quasi-equation* $\tau \doteq_s \tau \Rightarrow \tau_1 \doteq_s \tau_2$*, for an arbitrary* $\tau \in |\mathbf{T}|_s$*.*

Notice that equations and quasi-equations are not elements of a term algebra. Moreover, note that we have used $\doteq$ instead of $=$ in the definition of equations and quasi-equations since $=$, as a symbol, is part of the language of DAL. We define below when equations and quasi-equation are satisfied in an algebra.

Definition 5. *An equation* $\tau_1 \doteq_s \tau_2$ *is satisfied in an algebra* $\mathbf{A}$ *under an interpretation* h *iff* $h_s(\tau_1) = h_s(\tau_2)$*; and it is valid in* $\mathbf{A}$*, written* $\mathbf{A} \models \tau_1 \doteq \tau_2$*, iff* $\mathbf{A}, h \models \tau_1 \doteq_s \tau_2$ *for all interpretations* h *on* $\mathbf{A}$*. In turn, a quasi-equation* $\tau_1 \doteq_s \tau_2 \Rightarrow \tau_1' \doteq_{s'} \tau_2'$ *is satisfied in* $\mathbf{A}$ *under* h *iff* $h_s(\tau_1) = h_s(\tau_2)$ *implies* $h_s(\tau_1') = h_s(\tau_2')$*; and it is valid in* $\mathbf{A}$ *iff for any interpretation* h *on* $\mathbf{A}$*,* $\tau_1 \doteq_s \tau_2 \Rightarrow \tau_1' \doteq_{s'} \tau_2'$ *is satisfied in* $\mathbf{A}$ *under* h*.*

Just like equations give rise to classes of algebras called *varieties*, quasi-equations give rise to classes of algebras called *quasi-varieties*.

Definition 6. *A quasi-variety is the class of all algebras validating a set of quasi-equations; i.e., the class of all algebras where all quasi-equations in the set is valid.*

The final fundamental tool in the algebraic characterization of a logic is that of a congruence relation. When appropriately defined on term algebras, a congruence relation provides a method for constructing well-behaved algebras as quotient algebras out of "syntax". More formally, canonical algebraic models are obtained by taking the quotient of the term algebra using specific congruences.

Definition 7. *Let* $\mathbf{A}$ *be an algebra, and* $\cong\ = \{ \cong_s \subseteq |\mathbf{A}|_s \times |\mathbf{A}|_s \mid s \in S \}$ *be an* S*-indexed family of binary relations. We say that* $\cong$ *is a congruence on* $\mathbf{A}$ *iff every* $\cong_s \in \cong$ *is an equivalence relations on* $|\mathbf{A}|_s$*, and for every* $f \in \Omega_{s_1 \ldots s_n s}$*, it follows that* $a_i \cong_{s_i} a'_i$ *implies* $f_{\mathbf{A}}(a_1 \ldots a_n) \cong_s f_{\mathbf{A}}(a'_1 \ldots a'_n)$*. The quotient of* $\mathbf{A}$ *under* $\cong$ *is an algebra* $\mathbf{A}/\cong$ *where:* $|\mathbf{A}/\cong|_s = |\mathbf{A}|_s/\cong_s$*; and* $f_{(\mathbf{A}/\cong)}([a_1]_{\cong_{s_1}} \ldots [a_n]_{\cong_{s_n}}) = [f_{\mathbf{A}}(a_1 \ldots a_n)]_{\cong_s}$*.*

We conclude this section with a presentation of two classes of algebras we use in our algebraization of DAL: Boolean and Heyting algebras. We follow [18] and reach these particular algebras via *bounded distributive lattices* (BDLs). BDLs are characterized by a set of equations common to Boolean and Heyting algebras. This gives us a way to introduce concepts pertaining Boolean and Heyting algebras simultaneously. We introduce Heyting algebras as an extension of BDLs, and Boolean algebras as an extension of Heyting algebras. To keep the notation uncluttered, unless it is strictly necessary, we will omit the subscript $\mathbf{A}$ in the function $f_{\mathbf{A}}$ and simply write f. The context will always disambiguate whether we refer to the function or the symbol in the signature of $\mathbf{A}$.

Definition 8 ([18])**.** *Let* $\Lambda = \langle \{s\}, \{\{+, *\}_{sss}, \{0, 1\}_s\} \rangle$ *be a many-sorted signature. A BDL-algebra is a* Λ*-algebra* $\mathbf{L}$ *satisfying the following equations:*

L1. $\tau_1 + (\tau_2 + \tau_3) \doteq (\tau_1 + \tau_2) + \tau_3$	*L7.* $\tau_1 * (\tau_2 * \tau_3) \doteq (\tau_1 * \tau_2) * \tau_3$
L2. $\tau_1 + \tau_2 \doteq \tau_2 + \tau_1$	*L8.* $\tau_1 * \tau_2 \doteq \tau_2 * \tau_1$
L3. $\tau_1 + \tau_1 \doteq \tau_1$	*L9.* $\tau_1 * \tau_1 \doteq \tau_1$
L4. $\tau_1 + (\tau_1 * \tau_2) \doteq \tau_1$	*L10.* $\tau_1 * (\tau_1 + \tau_2) \doteq \tau_1$
L5. $\tau_1 + (\tau_2 * \tau_3) \doteq (\tau_1 + \tau_2) * (\tau_1 + \tau_3)$	*L11.* $\tau_1 * (\tau_2 + \tau_3) \doteq (\tau_1 * \tau_2) + (\tau_1 * \tau_3)$
L6. $\tau_1 + 1 \doteq 1$	*L12.* $\tau_1 * 0 \doteq 0$.

In Def. 8, τ_i is an element of the term algebra of type Λ. We use $\mathbf{L} = \langle L, +, *, 0, 1 \rangle$ to indicate an arbitrary BDL-algebra. We write $\preccurlyeq$ for the partial order implicit in a BDL algebra, i.e., $a \preccurlyeq b$ iff $a + b = b$.

Another important concept in our algebraization of DAL is that of an ideal (or, dually, a filter). Intuitively, an ideal is an initial set closed by unions (while a filter is a final set closed by intersections). Ideals appear in the definition of DAL as inherent properties of the formalization of permission and prohibition on actions [8].

Definition 9. *An ideal in a BDL-algebra* $\mathbf{L} = \langle L, +, *, 0, 1 \rangle$ *is a subset* $I \subseteq L$ *s.t.: for all* $x, y \in I$*,* $x + y \in I$*, and for all* $x \in I$ *and* $y \in L$*,* $(x * y) \in I$*. Dual to ideals are filters. A filter is a subset* $F \subseteq L$ *s.t.: for all* $x, y \in F$*,* $(x * y) \in F$*, and for all* $x \in F$ *and* $y \in L$*,* $x + y \in F$*.*

sorts	operations			
	actions	formulas	equality	normative
$S = \{\mathsf{a}, \mathsf{f}\}$	$\Omega_{\mathsf{aaa}} = \{\sqcup, \sqcap\}$ $\Omega_{\mathsf{aa}} = \{\bar{\ }\}$ $\Omega_{\mathsf{a}} = \{\mathsf{0}, \mathsf{1}\}$	$\Omega_{\mathsf{fff}} = \{\vee, \wedge\}$ $\Omega_{\mathsf{ff}} = \{\neg\}$ $\Omega_{\mathsf{f}} = \{\bot, \top\}$	$\Omega_{\mathsf{aaf}} = \{=\}$	$\Omega_{\mathsf{af}} = \{\mathsf{P}, \mathsf{F}\}$

Figure 4: The Signature used in the algebraization of DAL.

We use BDL-algebras to present Heyting and Boolean algebras.

Definition 10 ([18])**.** *Let* $\mathrm{H} = \langle\{s\}, \{\{\multimap, +, *\}_{sss}, \{0, 1\}_s\}\rangle$ *be a many-sorted signature. A Heyting algebra is an* H*-algebra satisfying the equations L1–L12 in Def. 8 together with the following equations:*

H1. $\tau_1 * (\tau_1 \multimap \tau_2) \doteq \tau_2$
H2. $((\tau_1 * \tau_2) \multimap \tau_1) * \tau_3 \doteq \tau_3$
H3. $\tau_1 * (\tau_2 \multimap \tau_3) \doteq \tau_1 * ((\tau_1 * \tau_2) \multimap (\tau_1 * \tau_3))$.

In Def. 10, τ_i is an element of the term algebra of type H. We use $\mathbf{H} = \langle H, \multimap, +, *, 0, 1\rangle$ to indicate an arbitrary Heyting algebra. We use $\bar{\tau}$ as an abbreviation of $\tau \multimap 0$. This abbreviation gives rise to an operation $^{-} : H \to H$ defined as $\bar{x} = x \multimap 0$. We will sometimes use $\mathbf{H} = \langle H, +, *, ^{-}, 0, 1\rangle$ to indicate an arbitrary Heyting algebra; in these cases, we assume $\multimap$ implicitly.

Definition 11. *Boolean algebras are Heyting algebras that validate: (LEM)* $\tau + \bar{\tau} \doteq 1$.

We use $\mathbf{B} = \langle B, +, *, ^{-}, 0, 1\rangle$ to indicate an arbitrary Boolean algebra; and $\mathbf{2}$ to indicate the Boolean algebra of exactly two elements. A Boolean algebra is *concrete* iff it is a field of sets. Important Boolean algebras in our setting are those freely generated and finitely generated [19]. We use Stone's representation theorem [21]. If $\mathbf{B}$ is a Boolean algebra, we use $\mathsf{s}(\mathbf{B})$ for its isomorphic Stone space, and $\varphi_{\mathbf{B}} : \mathbf{B} \to \mathsf{s}(\mathbf{B})$ for the isomorphism.

3.2 Algebraizing Deontic Action Logic

We start the algebraization of DAL introducing its signature, i.e., the symbols needed to capture the language of the logic in an algebraic way.

Definition 12. *The signature of* DAL *is a tuple* $\Sigma = \langle S, \Omega\rangle$ *where:* $S = \{\mathsf{a}, \mathsf{f}\}$*; and* $\bigcup \Omega = \{\sqcup, \sqcap, ^{-}, \mathsf{0}, \mathsf{1}, \vee, \wedge, \neg, \bot, \top, =, \mathsf{P}, \mathsf{F}\}$*. The symbols in* $\bigcup \Omega$ *are further categorized into sets* Ω_{aaa}*,* Ω_{aa}*,* Ω_{a}*,* Ω_{fff}*,* Ω_{ff}*,* Ω_{f}*,* Ω_{aaf}*,* Ω_{af} *summarized in Fig. 4.*

In our discussion on the algebraization of DAL, we take $\Sigma = \langle S, \Omega\rangle$ to be as in Def. 12. Intuitively, the sort symbols a and f in S categorize actions and formulas, respectively. In turn, we think of Ω as containing symbols for operations on actions, operations on formulas, and (heterogeneous) operations from actions to formulas.

Definition 13. *The term algebra* $\mathbf{T}$ *for* DAL *uses the set* Act_0 *as the set of variables of sort* a*, and the empty set* $\varnothing$ *as the set of variables of sort* f*. We call this algebra the deontic action term algebra, or the algebraic language of* DAL*.*

The term algebra $\mathbf{T}$ in Def. 13 is interpreted over *deontic action algebras*. Deontic action algebras are to DAL what Boolean algebras are to Classical Propositional Logic, or what Heyting algebras are to Intuitionistic Propositional Logic. We provide the precise definition of a deontic action algebra in Def. 14.

Definition 14. *A deontic action algebra is an algebra* $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ *of type* Σ *where:*[1] $\mathbf{A} = \langle A, \sqcup, \sqcap, ^-, 0, 1 \rangle$ *and* $\mathbf{F} = \langle F, \vee, \wedge, \neg, \bot, \top \rangle$ *are Boolean algebras, and* $=$*,* P*, and* F*, satisfy the conditions below*

1. $\mathsf{P}(a \sqcup b) =_{\mathbf{F}} \mathsf{P}(a) \wedge \mathsf{P}(b)$
2. $\mathsf{F}(a \sqcup b) =_{\mathbf{F}} \mathsf{F}(a) \wedge \mathsf{F}(b)$
3. $\mathsf{P}(a) \wedge \mathsf{F}(a) =_{\mathbf{F}} (a = 0)$
4. $(a = b) \wedge \mathsf{P}(a) \preccurlyeq \mathsf{P}(b)$
5. $(a = b) \wedge \mathsf{F}(a) \preccurlyeq \mathsf{F}(b)$
6. $a =_{\mathbf{A}} b$ *iff* $(a = b) =_{\mathbf{F}} \top$*.*

Let $h : \mathbf{T} \to \mathbf{D}$ *be an interpretation. We use* $\mathbf{D}, h \models \tau_1 \doteq \tau_2$ *as alternative notation for* $h(\tau_1) = h(\tau_2)$*. In turn, let* $\mathbb{D}$ *indicate the class of all deontic action algebras. We use* $\mathbb{D} \models \tau_1 \doteq \tau_2$ *as the universal quantification of* $\models$ *to all deontic action algebras in* $\mathbb{D}$ *and all interpretations on these algebras; i.e.,* $\mathbb{D} \models \tau_1 \doteq \tau_2$ *iff* $\mathbf{D}, h \models \tau_1 \doteq \tau_2$*, for all* $\mathbf{D} \in \mathbb{D}$*, and all interpretations* $h : \mathbf{T} \to \mathbf{D}$*.*

The next two results are immediate.

Proposition 3. $\mathbb{D} \models \alpha \doteq_{\mathsf{a}} \beta$ *iff* $\mathbb{D} \models (\alpha = \beta) \doteq_{\mathsf{f}} \top$*.*

Proposition 4. *The class* $\mathbb{D}$ *of all deontic action algebras is a quasi-variety.*

Proof. It suffices to show that the conditions in the definition of a deontic action algebra can be captured by equations, or quasi-equations. The interesting cases are:

1. $\mathsf{P}(a \sqcup b) =_{\mathbf{F}} \mathsf{P}(a) \wedge \mathsf{P}(b)$ expressed as $\mathsf{P}(a \sqcup b) \doteq_{\mathsf{f}} \mathsf{P}(a) \wedge \mathsf{P}(b)$;
2. $\mathsf{P}(a) \wedge \mathsf{F}(a) =_{\mathbf{F}} =(a, 0)$ expressed as $\mathsf{P}(a) \wedge \mathsf{F}(a) \doteq_{\mathsf{f}} a = 0$;
3. $(a = b) \wedge \mathsf{P}(a) \preccurlyeq \mathsf{P}(b)$ expressed as $((a = b) \wedge \mathsf{P}(a)) \vee \mathsf{P}(b) \doteq_{\mathsf{f}} \mathsf{P}(b)$; and
4. $a =_{\mathbf{A}} b$ iff $(a = b) =_{\mathbf{F}} \top$ expressed as the quasi-equations $a \doteq_{\mathsf{a}} b \Rightarrow (a = b) \doteq_{\mathsf{f}} \top$, and $(a = b) \doteq_{\mathsf{f}} \top \Rightarrow a \doteq_{\mathsf{a}} b$. □

The definition of a deontic action algebra in Def. 14 draws on ideas and terminology from Pratt's dynamic algebras [22]. We present the general structure of a deontic action algebra in a form slightly different from the general treatment of many-sorted algebras in Sec. 3.1. In doing this we wish to highlight the modular nature of deontic action algebras. In Sec. 4, we leverage this modularity to introduce variants of DAL by considering different algebraic structures for each component of the deontic action algebra. Finally, notice that, as made clear in Prop. 4, our treatment of equality in the logic results in the class of deontic action algebras being a quasi-variety. The result in Prop. 3 tells us we can dispense from explicitly referring to equations on actions as they are also captured as particular equations on formulas via equality in the logic.

[1] We use $=$ as the function interpreting '$=$' in $\mathbf{F}$, and $=_{\mathbf{A}}$ and $=_{\mathbf{F}}$ as equality in $\mathbf{A}$ and $\mathbf{F}$, respectively.

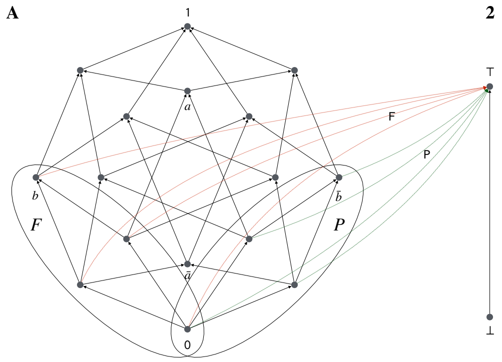

Figure 5: A Deontic Action Algebra.

Example 3. *Fig. 5 depicts the deontic action algebra* $\mathbf{D} = \langle \mathbf{A}, \mathbf{2}, =, \mathsf{P}, \mathsf{F} \rangle$ *where: the algebra* $\mathbf{A}$ *of actions is the free Boolean algebra on the set of generators* $\{a, b\}$*. In* $\mathbf{D}$*, the functions* P *and* F *are defined as:*

$$\mathsf{P}(x) = \begin{cases} \top & \text{if } x \preccurlyeq \bar{b} \\ \bot & \text{otherwise.} \end{cases} \qquad \mathsf{F}(x) = \begin{cases} \top & \text{if } x \preccurlyeq b \\ \bot & \text{otherwise.} \end{cases}$$

In Fig. 5, the elements of $|\mathbf{A}|$ *mapped to* $\top$ *by* P *and* F *are highlighted in green and red, respectively. To avoid cluttering the diagram, we omitted visual indications for the elements in* $|\mathbf{A}|$ *not mapped to* $\top$*. The sets* P *and* F *in the example represent the permitted and forbidden actions, respectively. Both* P *and* F *form ideals in* $\mathbf{A}$*, and their intersection contains only the element* $\mathsf{0}$*. From this example, it is easy to see that if* $\mathsf{P}(x) = \top$ *for all* $x \in |\mathbf{A}|$*, then* $\mathsf{F}(x) = \bot$ *for all* x *such that* $\mathsf{0} \prec x$*. Conversely, if* $\mathsf{F}(x) = \top$ *for all* $x \in |\mathbf{A}|$*, then* $\mathsf{P}(x) = \bot$ *for all* $\mathsf{0} \prec x$*. These two extreme cases are referred to as* deontic heaven *and* deontic hell*, respectively.*

Example 4. *Let* $\mathbf{D}$ *be the deontic action algebra in Fig. 5, and* $\mathsf{drinking}$, $\mathsf{driving}$*, and* $\mathsf{parking}$*, be basic action symbols. In addition, let* $h : \mathbf{T} \to \mathbf{D}$ *be an interpretation s.t.:* $h_{\mathsf{a}}(\mathsf{drinking}) = b$*,* $h_{\mathsf{a}}(\mathsf{driving}) = a$*, and* $h_{\mathsf{a}}(\mathsf{parking}) = \bar{a}$*. It follows that:*

1. $h(\overline{\mathsf{parking}} = \mathsf{driving}) =_{\mathbf{F}} \top$
2. $h(\mathsf{F}(\mathsf{drinking} \sqcap \mathsf{driving})) =_{\mathbf{F}} \top$
3. $h(\mathsf{P}(\mathsf{drinking} \sqcap \mathsf{parking})) =_{\mathbf{F}} \top$
4. $h(\mathsf{P}(\mathsf{driving} \sqcup \mathsf{parking})) \neq_{\mathbf{F}} \top$.

In brief, the deontic action algebra $\mathbf{D}$ *may be understood as the algebraic version of the deontic model* $\mathfrak{M}$ *in Sec. 2.*

The following proposition shows the ideals in the deontic action algebra in Fig. 5 are indeed a distinguishing characteristic of the operations of permission and prohibition.

Proposition 5. *Let* $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ *be a deontic action algebra. The pre-image* P *of* $\top$ *under* P*, as well as the preimage* F *of* $\top$ *under* F*, are ideals in* $\mathbf{A}$ *s.t.* $P \cap F = \{\mathsf{0}\}$*.*

Proof. The result is obtained from the following:

1. For all $\{a, b\} \subseteq P$, $a \sqcup b \in P$. The proof is as follows. Let $\{a, b\} \subseteq P$. Then, $\mathsf{P}(a) =_{\mathbf{F}} \mathsf{P}(b) =_{\mathbf{F}} \top$, and $\mathsf{P}(a) \wedge \mathsf{P}(b) =_{\mathbf{F}} \top$. The properties of P in Def. 14 ensure $\mathsf{P}(a) \wedge \mathsf{P}(b) =_{\mathbf{F}} \mathsf{P}(a \sqcup b)$. This implies $\mathsf{P}(a \sqcup b) =_{\mathbf{F}} \top$; and so $(a \sqcup b) \in P$.

2. For all $a \in P$ and $b \in |\mathbf{A}|$, $(a \sqcap b) \in P$. The proof is as follows. Let $a \in P$ and $b \in |\mathbf{A}|$. We know $\mathsf{P}(a) =_{\mathbf{F}} \top$ and $a =_{\mathbf{A}} a \sqcup (a \sqcap b)$. This means $\mathsf{P}(a \sqcup (a \sqcap b)) =_{\mathbf{F}} \top$. The properties of P in Def. 14 ensure $\mathsf{P}(a \sqcup (a \sqcap b)) =_{\mathbf{F}} \mathsf{P}(a) \wedge \mathsf{P}(a \sqcap b)$. This means, $\mathsf{P}(a) \wedge \mathsf{P}(a \sqcap b) =_{\mathbf{F}} \top$. From our supposition, $\mathsf{P}(a \sqcap b) =_{\mathbf{F}} \top$; and so $(a \sqcap b) \in P$.

3. The arguments in 1 and 2 remain true if we replace P and P for F and F, resp.

4. $P \cap F = \{0\}$. The proof is as follows. Note that $\mathsf{P}(0) =_{\mathbf{F}} \mathsf{F}(0) =_{\mathbf{F}} \top$; and so $\{0\} \subseteq P \cap F$. In turn, consider an arbitrary $a \in (P \cap F)$. Then, $\mathsf{P}(a) =_{\mathbf{F}} \mathsf{F}(a) =_{\mathbf{F}} \top$. This implies $\mathsf{P}(a) \wedge \mathsf{F}(a) =_{\mathbf{F}} \top$, and so $(a = 0) =_{\mathbf{F}} \top$. The 'iff' condition in Def. 14 ensures $a =_{\mathbf{A}} 0$. Since a is arbitrary, the last step tells us that any element in $P \cap F$ is equal to 0. Therefore, $P \cap F \subseteq \{0\}$. □

We proceed to connect the deontic action algebras in $\mathbb{D}$ with the theorems of DAL.

Theorem 6 (Soundness)**.** *If φ is a theorem of* DAL*, then,* $\mathbb{D} \models \varphi \doteq \top$*.*

Proof. Let $\mathbf{D} \in \mathbb{D}$ and $h : \mathbf{T} \to \mathbf{D}$ be any interpretation. We continue by induction on the length of the proof of φ. We prove the more interesting cases; others are similar.

1. $h_{\mathsf{f}}(\mathsf{P}(\alpha \sqcup \beta) \leftrightarrow (\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta))) =_{\mathbf{F}} \top$. The result follows from items (a)–(c) below.

   (a) $h_{\mathsf{f}}(\mathsf{P}(\alpha \sqcup \beta) \leftrightarrow (\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta))) =_{\mathbf{F}}$
   $h_{\mathsf{f}}((\neg \mathsf{P}(\alpha \sqcup \beta) \vee (\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta))) \wedge (\neg(\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta)) \vee \mathsf{P}(\alpha \sqcup \beta))) =_{\mathbf{F}}$
   $h_{\mathsf{f}}(\neg \mathsf{P}(\alpha \sqcup \beta) \vee (\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta))) \wedge h_{\mathsf{f}}((\neg(\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta)) \vee \mathsf{P}(\alpha \sqcup \beta)))$.

   (b) $h_{\mathsf{f}}(\neg \mathsf{P}(\alpha \sqcup \beta) \vee (\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta))) =_{\mathbf{F}}$
   $\neg \mathsf{P}(h_{\mathsf{a}}(\alpha \sqcup \beta)) \vee (\mathsf{P}(h_{\mathsf{a}}(\alpha)) \wedge \mathsf{P}(h_{\mathsf{a}}(\beta))) =_{\mathbf{F}}$
   $\neg \mathsf{P}(h_{\mathsf{a}}(\alpha \sqcup \beta)) \vee (\mathsf{P}(h_{\mathsf{a}}(\alpha) \sqcup h_{\mathsf{a}}(\beta))) =_{\mathbf{F}}$ .... Def. 14(1)
   $\neg \mathsf{P}(h_{\mathsf{a}}(\alpha \sqcup \beta)) \vee (\mathsf{P}(h_{\mathsf{a}}(\alpha \sqcup \beta))) =_{\mathbf{F}} \top$.

   (c) $h_{\mathsf{f}}(\neg(\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta)) \vee \mathsf{P}(\alpha \sqcup \beta)) =_{\mathbf{F}} \top$ is similar to (b).

2. $h_{\mathsf{f}}((\mathsf{P}(\alpha) \wedge \mathsf{F}(\alpha)) \to (\alpha = 0)) =_{\mathbf{F}} \top$. Then,

   $h_{\mathsf{f}}((\mathsf{P}(\alpha) \wedge \mathsf{F}(\alpha)) \to (\alpha = 0)) =_{\mathbf{F}}$
   $h_{\mathsf{f}}(\neg(\mathsf{P}(\alpha) \wedge \mathsf{F}(\alpha)) \vee (\alpha = 0)) =_{\mathbf{F}}$
   $\neg(\mathsf{P}(h_{\mathsf{a}}(\alpha)) \wedge \mathsf{F}(h_{\mathsf{a}}(\alpha))) \vee h_{\mathsf{f}}(\alpha = 0) =_{\mathbf{F}}$
   $\neg(\mathsf{P}(h_{\mathsf{a}}(\alpha)) \wedge \mathsf{F}(h_{\mathsf{a}}(\alpha))) \vee h_{\mathsf{f}}(\alpha = 0) =_{\mathbf{F}}$
   $\neg(h_{\mathsf{a}}(\alpha) = 0) \vee h_{\mathsf{f}}(\alpha = 0) =_{\mathbf{F}}$ .... Def. 14(3)
   $\neg h_{\mathsf{f}}(\alpha = 0) \vee h_{\mathsf{f}}(\alpha = 0) =_{\mathbf{F}} \top$.

3. $h_{\mathsf{f}}(((\alpha = \beta) \wedge \mathsf{P}(\alpha)) \to \mathsf{P}(\beta)) =_{\mathbf{F}} \top$. Then,

   $h_{\mathsf{f}}(((\alpha = \beta) \wedge \mathsf{P}(\alpha)) \to \mathsf{P}(\beta)) =_{\mathbf{F}}$
   $\neg h_{\mathsf{f}}((\alpha = \beta) \wedge \mathsf{P}(\alpha)) \vee \mathsf{P}(h_{\mathsf{a}}(\beta)) =_{\mathbf{F}}$
   $\neg h_{\mathsf{f}}((\alpha = \beta) \wedge \mathsf{P}(\alpha)) \vee ((h_{\mathsf{a}}(\alpha) = h_{\mathsf{a}}(\beta)) \wedge \mathsf{P}(h_{\mathsf{a}}(\alpha))) \vee \mathsf{P}(h_{\mathsf{a}}(\beta)) =_{\mathbf{F}}$ .... Def. 14(4)
   $\neg h_{\mathsf{f}}((\alpha = \beta) \wedge \mathsf{P}(\alpha)) \vee h_{\mathsf{f}}((\alpha = \beta) \wedge \mathsf{P}(\alpha)) \vee \mathsf{P}(h_{\mathsf{a}}(\beta)) =_{\mathbf{F}} \top$.

The result in (3.) is a particular case of the axioms E2 in Fig. 3. Other instances can be proven by induction on the size of the formula φ. $\square$

Thm. 6 implies that there are formulas of DAL which are not provable in the logic. In particular, negations of theorems are not provable. To see why, consider a theorem φ. Now, let $\mathbf{D} = \langle \mathbf{A}, \mathbf{2}, =, \mathsf{P}, \mathsf{F} \rangle$, and $h : \mathbf{T} \to \mathbf{D}$ be any interpretation on this deontic action algebra. From Thm. 6, we have $h_{\mathsf{f}}(\varphi) = \top$. Since h is a homomorphism, $h_{\mathsf{f}}(\neg\varphi) = \bot$. Using the contrapositive of Thm. 6, $\neg\varphi$ is not a theorem; i.e., it is not provable.

The converse of Thm. 6, i.e., the algebraic completeness of DAL, requires us to show that every non-theorem φ of DAL is falsified in some deontic action algebra $\mathbf{D}$ (i.e., there is an interpretation $h : \mathbf{T} \to \mathbf{D}$ s.t. $h_{\mathsf{f}}(\varphi) \neq \top$). We arrive at this result introducing an appropriate notion of congruence, and constructing a quotient algebra via this congruence.

Proposition 7. *Let $\mathbf{T}$ be the deontic term algebra, and $\cong_{\mathsf{a}} \subseteq |\mathbf{T}|_{\mathsf{a}} \times |\mathbf{T}|_{\mathsf{a}}$ and $\cong_{\mathsf{f}} \subseteq |\mathbf{T}|_{\mathsf{f}} \times |\mathbf{T}|_{\mathsf{f}}$ be s.t.: 1. $\alpha \cong_{\mathsf{a}} \beta$ iff $\alpha = \beta$ is a theorem, and 2. $\varphi \cong_{\mathsf{f}} \psi$ iff $\varphi \leftrightarrow \psi$ is a theorem. It follows that $\cong_{\mathsf{a}}$ and $\cong_{\mathsf{f}}$ define a congruence $\cong$ on $\mathbf{T}$.*

Proposition 8. *The quotient of the deontic action term algebra $\mathbf{T}$ under $\cong$ is a structure $\mathbf{L} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ where: 1. $|\mathbf{A}| = \mathsf{Act}/{\cong_{\mathsf{a}}}$, 2. $|\mathbf{F}| = \mathsf{Form}/{\cong_{\mathsf{f}}}$, and 3. the operations in $\mathbf{L}$ are those induced by the equivalence classes in $\cong$. It follows that $\mathbf{L} \in \mathbb{D}$.*

Proof. It is clear that $\mathbf{A}$ and $\mathbf{F}$ are Boolean algebras. Let us use $\check{\_}$ to indicate the operations in $\mathbf{L}$ induced by $\cong$, and to separate them from the corresponding symbols. The result is concluded if $\check{=}$, $\check{\mathsf{P}}$, and $\check{\mathsf{F}}$ satisfy the conditions in Def. 14. We prove some interesting cases only.

1. $\check{\mathsf{P}}([\alpha \sqcup \beta]_{\cong_{\mathsf{a}}}) =_{\mathbf{F}} \check{\mathsf{P}}([\alpha]_{\cong_{\mathsf{a}}}) \wedge \check{\mathsf{P}}([\beta]_{\cong_{\mathsf{a}}})$

   $\check{\mathsf{P}}([\alpha \sqcup \beta]_{\cong_{\mathsf{a}}}) =_{\mathbf{F}} [\mathsf{P}(\alpha \sqcup \beta)]_{\cong_{\mathsf{f}}} =_{\mathbf{F}}$
   $[\mathsf{P}(\alpha) \wedge \mathsf{P}(\beta)]_{\cong_{\mathsf{f}}} =_{\mathbf{F}}$ ...... Fig. 3(D1)
   $[\mathsf{P}(\alpha)]_{\cong_{\mathsf{f}}} \check{\wedge} [\mathsf{P}(\beta)]_{\cong_{\mathsf{f}}} =_{\mathbf{F}} \check{\mathsf{P}}([\alpha]_{\cong_{\mathsf{a}}}) \check{\wedge} \check{\mathsf{P}}([\beta]_{\cong_{\mathsf{a}}})$.

2. $\check{\mathsf{P}}([\alpha]_{\cong_{\mathsf{a}}}) \check{\wedge} \check{\mathsf{F}}([\alpha]_{\cong_{\mathsf{a}}}) =_{\mathbf{F}} [\alpha]_{\cong_{\mathsf{a}}} \check{=} \mathsf{0}$

   $\check{\mathsf{P}}([\alpha]_{\cong_{\mathsf{a}}}) \check{\wedge} \check{\mathsf{F}}([\alpha]_{\cong_{\mathsf{a}}}) =_{\mathbf{F}} [\mathsf{P}(\alpha)]_{\cong_{\mathsf{f}}} \check{\wedge} [\mathsf{F}(\alpha)]_{\cong_{\mathsf{f}}} =_{\mathbf{F}}$
   $[\mathsf{P}(\alpha) \wedge \mathsf{F}(\alpha)]_{\cong_{\mathsf{f}}} =_{\mathbf{F}}$
   $[\alpha = \mathsf{0}]_{\cong_{\mathsf{f}}} =_{\mathbf{F}}$ ...... Fig. 3(D3)
   $[\alpha]_{\cong_{\mathsf{a}}} \check{=} \mathsf{0}$.

3. $[\alpha]_{\cong_{\mathsf{a}}} =_{\mathbf{A}} [\beta]_{\cong_{\mathsf{a}}}$ iff $[\alpha]_{\cong_{\mathsf{a}}} \check{=} [\beta]_{\cong_{\mathsf{a}}} =_{\mathbf{F}} \top$.

   Left-to-right: Let $[\alpha]_{\cong_{\mathsf{a}}} =_{\mathbf{A}} [\beta]_{\cong_{\mathsf{a}}}$. This assumption implies, by definition, that $\alpha = \beta$ is a theorem. Immediately, $(\alpha = \beta) \leftrightarrow \top$ is also a theorem. But this means, $[\alpha = \beta]_{\cong_{\mathsf{f}}} =_{\mathbf{F}} \top$. Thus, $[\alpha]_{\cong_{\mathsf{a}}} \check{=} [\beta]_{\cong_{\mathsf{a}}} =_{\mathbf{F}} \top$.

   Right-to-left: Similarly, let $[\alpha]_{\cong_{\mathsf{a}}} \check{=} [\beta]_{\cong_{\mathsf{a}}} =_{\mathbf{F}} \top$. Then, $[\alpha = \beta]_{\cong_{\mathsf{f}}} = \top$. This means $\alpha = \beta$ is a theorem. And so, $[\alpha]_{\cong_{\mathsf{a}}} =_{\mathbf{A}} [\beta]_{\cong_{\mathsf{a}}}$. $\square$

We call the quotient algebra $\mathbf{L}$ in Prop. 8 the Lindenbaum-Tarski deontic action algebra. In brief, $\mathbf{L}$ is a canonical algebra that captures theoremhood in DAL. From this observation, we obtain the following result.

Theorem 9 (Completeness). $\mathbb{D} \models \varphi \doteq \top$ *implies* φ *is a theorem.*

Proof. We show that if φ is not a theorem, then $\mathbb{D} \not\models \varphi \doteq \top$. Let φ be a non-theorem. From the definition of $\cong$, $[\varphi]_{\cong_\mathsf{f}} \neq_{\mathbf{F}} \top$. Define a function $h : \mathsf{Act}_0 \to \mathbf{A}/\cong$ that sends each $\mathsf{a}_i \in \mathsf{Act}_0$ to the equivalence class $[\mathsf{a}_i]_{\cong_\mathsf{a}}$. The function h extends uniquely to an interpretation $\check{h} : \mathbf{T} \to \mathbf{L}$ such that $\check{h}(\varphi) =_{\mathbf{F}} [\varphi]_{\cong_\mathsf{f}}$. Therefore, $\mathbb{D} \not\models \varphi \doteq \top$. $\square$

Corollary 10. $\mathbb{D} \models \varphi \doteq \top$ *implies* φ *is a tautology.*

Proof. Immediate from Thms. 2 and 9. $\square$

3.3 Deontic Action Algebras and Deontic Action Models

Interestingly, the algebraization of DAL enjoys a Stone-type representation result connecting the algebraic semantics using deontic action algebras with the original semantics using deontic action models. This connection provides us with another completeness result for the theorems of DAL.

Recall that Stone's representation theorem establishes that every Boolean algebra is isomorphic to a field of sets [21]. Such a result reveals a tight connection between the properties of an abstract structure with those of a *concrete* one (a collection of sets). This is also true for deontic action algebras. We begin by introducing the definition of a concrete deontic action algebra.

Definition 15. *A deontic action algebra* $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ *is* concrete *iff* $\mathbf{A}$ *and* $\mathbf{F}$ *are fields of sets. Let* $\mathbb{D}(0)$ *be the class of concrete deontic algebras. For equations of the appropriate sort, we use* $\mathbb{D}(0) \models \tau_1 \doteq \tau_2$ *as the analogue of* $\mathbb{D} \models \tau_1 \doteq \tau_2$ *in Def. 14.*

We prove that validity in deontic action algebras reduces to validity in concrete deontic algebras. In this way, concrete deontic algebras enable us to connect the algebraic semantics of DAL with Segerberg's original semantics via Stone's duality.

Theorem 11. $\mathbb{D}(0) \models \varphi \doteq \top$ *iff* $\mathbb{D} \models \varphi \doteq \top$.

Proof. The right-to-left direction is straightforward. The proof for the left-to-right direction is by contrapositive. Assume that $\mathbb{D} \not\models \varphi \doteq \top$. This means that we have a deontic action algebra $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ and an interpretation $h : \mathbf{T} \to \mathbf{D}$ s.t. $h_\mathsf{f}(\varphi) \neq_{\mathbf{F}} \top$. Via the Stone duality result for Boolean algebras, we can construct a concrete deontic action algebra $\mathbf{D}' = \langle \mathbf{A}', \mathbf{F}', =', \mathsf{P}', \mathsf{F}' \rangle$ that is isomorphic to $\mathbf{D}$. Moreover, we can define an interpretation $h' : \mathbf{T} \to \mathbf{D}'$ s.t. $h'(a_i) = \varphi_{\mathbf{A}'}(h(a_i))$ (with $\varphi_{\mathbf{A}'}$ being the Stone isomorphism for $\mathbf{A}'$). This construction ensures $h'(\varphi) \neq_{\mathbf{F}'} \top$, therefore $\mathbb{D}(0) \not\models \varphi \doteq \top$. $\square$

We can now link deontic action models with concrete deontic action algebras.

Definition 16. *Let* $\mathfrak{M} = \langle E, P, F \rangle$ *be a deontic action model,* $v : \mathsf{Act}_0 \to 2^E$ *be a valuation on* $\mathfrak{M}$*, and* $A = \{ v(\mathsf{a}_i) \mid \mathsf{a}_i \in \mathsf{Act}_0 \}$*. Define a concrete deontic action algebra* $\mathrm{alg}(\mathfrak{M}, v) = \langle \mathbf{A}, \mathbf{2}, =, \mathsf{P}, \mathsf{F} \rangle$ *where:*

$$\mathbf{A} = \langle 2^A, \cup, \cap, {}^\complement, \varnothing, A \rangle \qquad (a = b) =_{\mathbf{2}} \top \textit{ iff } a =_{\mathbf{A}} b \qquad \begin{array}{l} \mathsf{P}(a) = \top \textit{ iff } a \subseteq P \\ \mathsf{F}(a) = \top \textit{ iff } a \subseteq F. \end{array}$$

Define also the interpretation $h : \mathbf{T} \to \mathrm{alg}(\mathfrak{M}, v)$ *as the unique extension of* v.

Similarly, concrete deontic algebras give rise to deontic action models.

Definition 17. *Let* $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ *be a concrete deontic algebra,* $h : \mathbf{T} \to \mathbf{D}$ *be an interpretation. Define a deontic action model* $\mathrm{mod}(\mathbf{D}, h) = \langle E, P, F \rangle$ *where:*

$$E = |\mathbf{A}| \qquad P = \bigcup \{ a \mid \mathsf{P}(a) =_{\mathbf{F}} \top \} \qquad F = \bigcup \{ a \mid \mathsf{F}(a) =_{\mathbf{F}} \top \}.$$

Define also a valuation v *on* $\mathrm{mod}(\mathbf{D}, f)$ *as the restriction of* h *to* Act_0.

If seen as operators, mod and alg are inverses of each other.

Theorem 12. $\mathrm{alg}(\mathrm{mod}(\mathbf{D}, v), h) = \mathbf{D}$ *and* $\mathrm{mod}(\mathrm{alg}(\mathfrak{M}, v), h) = \mathfrak{M}$.

In light of Thm. 12, we obtain the following result.

Corollary 13. *It follows that:* $\mathfrak{M}, v \Vdash \varphi$ *iff* $\mathrm{alg}(\mathfrak{M}, v), h \models \varphi \doteq \top$*; and* $\mathbf{D}, h \models \varphi \doteq \top$ *iff* $\mathrm{mod}(\mathbf{D}, h), v \models \varphi$.

The results in Thm. 12 and cor. 13 enable us to prove the completeness of DAL w.r.t. Segerberg's original deontic models entirely in an algebraic way.

Theorem 14. *A formula* φ *is a theorem iff it is a tautology.*

Proof. Suppose that φ is a theorem. From Cor. 10, $\mathbb{D} \models \varphi \doteq \top$. From Thm. 11, $\mathbb{D}(0) \models \varphi \doteq \top$. From Cor. 13, φ is a tautology. Thus, φ is a theorem implies φ is a tautology. Using these results in the inverse order we obtain φ is a tautology implies φ is a theorem. □

4 Other Deontic Action Logics

One of the main benefits of our algebraic treatment of DAL is that it can be modified using standard algebraic tools to cope with different versions of the logic. In this section, firstly, we show how classical variations of DAL can be algebraically captured by standard algebraic constructions, that is, equations, sub-algebras, and generated algebras. One interesting point of these extensions is that the soundness and completeness properties of these extensions can be obtained by applying similar constructions to the Lindenbaum algebra presented in earlier sections. Secondly, we consider intuitionistic versions of the logic by replacing the Boolean components of the algebras by Heyting algebras. As far as we are aware, intutionistic deontic action logics have not been considered before in the literature.

4.1 Previously Proposed Variants of DAL

Segerberg's foundational contributions [8] laid the groundwork for a family of closely related deontic action logics. Building on this foundation, the five systems introduced in [14] are particularly interesting as they address specific open issues in the field of Deontic Logic —such as the *principle of deontic closure*. We show how our algebraic framework can be easily extended to characterize each of these logics, showcasing the adaptability and versatility of deontic action algebras. In the rest of this section, we use a to indicate a basic action symbol and a to indicate its corresponding interpretation in an algebra.

The first of the five systems in [14], here called $\mathsf{DAL}(1)$, is obtained from DAL by adding the set $\{\, \mathsf{F}(\mathsf{a}) \vee \mathsf{P}(\mathsf{a}) \mid \mathsf{a} \in \mathsf{Act}_0 \,\}$ of formulas as additional axioms. Intuitively, this new set of axioms aims to capture the so-called *principle of deontic closure*—what is not forbidden is permitted—at the level of basic actions (i.e., action generators). The algebraic counterpart of $\mathsf{DAL}(1)$ is determined by the class of deontic action algebras whose algebra of actions is generated by a set of generators s.t. the condition $\mathsf{F}(a) \vee \mathsf{P}(a) =_{\mathbf{F}} \top$ holds for every generator a.

The second system, here called $\mathsf{DAL}(2)$, is obtained from $\mathsf{DAL}(1)$ by adding the formula $\mathsf{P}(\bar{\mathsf{a}}_0 \sqcap \cdots \sqcap \bar{\mathsf{a}}_n) \vee \mathsf{F}(\bar{\mathsf{a}}_0 \sqcap \cdots \sqcap \bar{\mathsf{a}}_n)$ as an additional axiom of the logic, under the proviso that $\mathsf{Act}_0 = \{\mathsf{a}_0, \ldots, \mathsf{a}_n\}$ for some $n \in \mathbb{N}_0$; i.e., under the proviso that there are finitely many basic action symbols. Intuitively, this additional axiom states that not performing any of the basic actions is permitted or forbidden. The algebraic counterpart of $\mathsf{DAL}(2)$ corresponds to the class of deontic action algebras with a finitely generated atomic Boolean algebra of actions $\mathbf{A}$ satisfying the condition $\mathsf{P}(\bar{a}_1 \sqcap \cdots \sqcap \bar{a}_n) \vee \mathsf{F}(\bar{a}_1 \sqcap \cdots \sqcap \bar{a}_n) =_{\mathbf{F}} \top$ for $\{a_0, \ldots, a_n\}$ the set of generators of $\mathbf{A}$.

The third system, $\mathsf{DAL}(3)$, is obtained from $\mathsf{DAL}(2)$ by adding $(\mathsf{a}_0 \sqcup \cdots \sqcup \mathsf{a}_n) = \mathsf{1}$ as an additional axiom. Intuitively, this new axiom indicates that the agent can only perform actions in $\{\mathsf{a}_1, \ldots, \mathsf{a}_n\}$. The algebraic counterpart of $\mathsf{DAL}(3)$ corresponds to the subclass of $\mathsf{DAL}(2)$ further satisfying the condition $a_0 \sqcup \cdots \sqcup a_n = 1$.

The fourth system, $\mathsf{DAL}(4)$, aims to capture the principle of deontic closure at the level of "atomic" actions. Formally, the language of the logic assumes a finite set $\{\mathsf{a}_0, \ldots, \mathsf{a}_n\}$ of basic action symbols. Its axiomatization adds all the formulas in $\{\, \mathsf{P}(\tilde{\mathsf{a}}_0 \sqcap \cdots \sqcap \tilde{\mathsf{a}}_n) \vee \mathsf{F}(\tilde{\mathsf{a}}_0 \sqcap \cdots \sqcap \tilde{\mathsf{a}}_n) \mid \tilde{\mathsf{a}}_i \in \{\mathsf{a}_i, \bar{\mathsf{a}}_i\} \,\}$ as additional axioms to DAL. The algebraic counterpart of $\mathsf{DAL}(4)$ corresponds to the class of all deontic action algebras with a finitely generated and atomic algebra of actions, whose atoms a satisfy the condition $\mathsf{P}(a) \vee \mathsf{F}(a) =_{\mathbf{F}} \top$.

The fifth and last system in [14], here called $\mathsf{DAL}(5)$, is the union of $\mathsf{DAL}(3)$ and $\mathsf{DAL}(4)$. Naturally, its algebraic counterpart corresponds to the intersection of the classes of deontic action algebras characterizing $\mathsf{DAL}(3)$ and $\mathsf{DAL}(4)$.

We now present soundness and completeness results of each of these logics. To this end, we introduce the auxiliary definitions of *deontic action subalgebra* and *deontic action generated algebra*. Both are analogous to the standard case.

Definition 18. *Let* $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ *and* $\mathbf{D}' = \langle \mathbf{A}', \mathbf{F}', =', \mathsf{P}', \mathsf{F}' \rangle$ *be two deontic action algebras, we say that* $\mathbf{D}'$ *is a subalgebra of* $\mathbf{D}$ *iff: 1.* $\mathbf{A}'$ *is a subalgebra of* $\mathbf{A}$*; 2.* $\mathbf{F}'$ *is a subalgebra of* $\mathbf{F}$*; and 3.* $='$, F', *and* P' *are restrictions of* $=$, F, *and* P *to* $\mathbf{A}'$ *and* $\mathbf{F}'$*, respectively.*

Definition 19. *Let* $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ *be a deontic action algebra. In addition, let* $A' \subseteq |\mathbf{A}|$ *and* $F' \subseteq |\mathbf{F}|$*. The sets* A' *and* F' *are called generators. The deontic action algebra generated by* A' *and* F' *is the subalgebra* $\mathbf{D} = \langle \mathbf{A}', \mathbf{F}', =', \mathsf{P}', \mathsf{F}' \rangle$ *of* $\mathbf{D}$ *where: 1.* $\mathbf{A}'$ *is the intersection of all the subalgebras of* $\mathbf{A}$ *whose carrier set contains* A'*; and 2.* $\mathbf{F}'$ *is the intersection of all the subalgebras of* $\mathbf{F}$ *whose carrier set contains* F'*.*

The following theorem extends Thm. 9 for DAL to its variants $\mathsf{DAL}(i)$.

Theorem 15. *φ is a theorem of $\mathsf{DAL}(i)$ iff $\mathbb{D}(i) \vdash \varphi \doteq \top$.*

Proof. The proof is direct extension of that in Thm. 9. We only sketch relevant steps.

Soundness. For $\mathsf{DAL}(1)$ we need to show for all $\mathbf{D} \in \mathbb{D}(1)$ and all interpretations $h : \mathbf{T} \to \mathbf{D}$, that $h(\mathsf{F}(\mathsf{a}_i) \vee \mathsf{P}(\mathsf{a}_i)) =_{\mathbf{F}} \top$. Then:

$$h(\mathsf{F}(\mathsf{a}_i) \vee \mathsf{P}(\mathsf{a}_i)) =_{\mathbf{F}} h(\mathsf{F}(\mathsf{a}_i)) \vee h(\mathsf{P}(\mathsf{a}_i)) =_{\mathbf{F}} \mathsf{F}(h(\mathsf{a}_i)) \vee \mathsf{P}(h(\mathsf{a}_i)) =_{\mathbf{F}} \top.$$

Note that for every $a_i \in \mathsf{Act}_0$, $h(a_i)$ is a generator, and that homomorphisms between generated Boolean algebras are determined by the mapping between their generators. For the other variants the proofs are similar using the properties of generators, homomorphisms, and the new equations for each case.

Completeness. Similarly to our result in Thm. 9, for each $\mathsf{DAL}(i)$, we need to define an equivalent to the Lindenbaum-Tarski Algebra of DAL. We describe the procedure for $\mathsf{DAL}(1)$. The other cases use the same argument. First, consider the Lindenbaum-Tarski $\mathbf{L}$ in Prop. 8, and consider the subalgebra $\mathbf{L}(1)$ generated by the generators $A' = \{ [\mathsf{a}_i]_{\cong_\mathsf{a}} \mid \mathsf{a}_i \in \mathsf{Act}_0 \}$, and $F' = \mathsf{Form}/_{\cong_\mathsf{f}}$. Furthermore, consider the congruence $\cong_{(1)}$ over $\mathbf{L}(1)$ induced by theoremhood in $\mathsf{DAL}(1)$, i.e., the axioms of DAL plus the new axiom set $\{ \mathsf{F}(\mathsf{a}_i) \vee \mathsf{P}(\mathsf{a}_i) \mid \mathsf{a}_i \in \mathsf{Act}_0 \}$. From its construction, $\mathbf{L}(1)/\cong_{(1)}$ is such that $\mathsf{F}([\mathsf{a}_i]_{\cong_\mathsf{a}}) \vee \mathsf{P}([\mathsf{a}_i]_{\cong_\mathsf{a}}) = \top$. This algebra provides the canonical deontic action algebra for $\mathsf{DAL}(1)$. The proof for $\mathsf{DAL}(i)$, for $i \in \{2, 3, 4, 5\}$, can be obtained by a similar procedure: a subalgebra of the original Lindenbaum Algebra is considered, this subalgebra is quotiented by the corresponding axioms, obtaining an algebra that allows us to prove the completenes for the corresponding version of the logic. □

4.2 Introducing Propositions

Our algebraization of DAL features an unusual characteristic: the use of an empty set of variables of sort f in the definition of the term algebra $\mathbf{T}$ in Def. 13. A more natural approach is to consider a countable set Prop of proposition symbols as variables of sort f, analogous to the set Act_0 of basic action symbols. Incorporating the set Prop into DAL results in a new deontic action logic, which we denote as $\mathsf{DAL}(\mathsf{Prop})$. The construction of this new logic is relatively direct, as are its soundness and completeness results. Interestingly also, we show that this new logic has certain advantages over DAL for modeling scenarios that require explicit propositional reasoning. Including propositions also resonates with insights from the literature. In particular, [**?**] combines deontic systems for actions and situations into a unified framework. In the same spirit, we show how deontic action algebras can be enriched with an algebra of propositions, thereby enabling a more expressive treatment of action/situation scenarios.

Deontic Action Algebras and Propositions.

The logic $\mathsf{DAL}(\mathsf{Prop})$ extends the language of DAL by incorporating propositional symbols from the set $\mathsf{Prop} = \{ p_i \mid i \in \mathbb{N}_0 \}$ as base cases in the recursive definition of formulas. The axiom system for this logic is directly obtained from that of DAL, accounting for this enriched notion of formula. The algebraization of $\mathsf{DAL}(\mathsf{Prop})$ retains the same signature as DAL. Its algebraic language is the term algebra $\mathbf{T}_1$, built over two disjoint sets of variables: Act_0, representing variables of sort a (actions), and Prop, representing variables of sort f (formulas). Interpretations of $\mathbf{T}_1$ are as in Sec. 3.1. Specifically, given a deontic action algebra $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$, an interpretation is a homomorphism $h : \mathbf{T}_1 \to \mathbf{D}$. We derive the soundness and completeness of $\mathsf{DAL}(\mathsf{Prop})$ by adapting the corresponding results for DAL in Sec. 3.

Theorem 16. *φ is a theorem of* $\mathsf{DAL}(\mathsf{Prop})$ *iff* $\mathbb{D} \models \varphi \doteq \top$.

Proof. The proof follows the steps of that of Thm. 6, we highlight some subtle details.

Soundness. The proof of soundness is, as in Thm. 6, by induction on the length of proofs. Note that the set of proofs is defined as in Thm. 6, but instantiations of axiom schemas may now contain proposition symbols. E.g., $(p \wedge \neg p) \leftrightarrow \perp$ is an instance of an axiom schema, and a theorem. This does not affect the proof given in Thm. 6, which is immediately lifted to a proof of soundness for $\mathsf{DAL}(\mathsf{Prop})$.

Completeness. For completeness, we need to redefine the Lindenbaum algebra. First, in this case, we consider the term algebra $\mathbf{T}_1$ that contains also formulas with propositions. The congruence $\cong$ in Prop. 8 is used to construct the quotient algebra. This quotient algebra is a deontic action algebra. Note that this algebra also contains formula terms with propositions by definition. Finally, adapting the proof of Thm. 9 we obtain the algebraic completeness result. ☐

In summary, $\mathsf{DAL}(\mathsf{Prop})$ differs only from DAL in their respective term algebras. That is, while DAL is associated with a term algebra $\mathbf{T}$ built over an empty set of variables of sort f, the term algebra $\mathbf{T}_1$ associated to $\mathsf{DAL}(\mathsf{Prop})$ uses the set Prop as the set of variables of sort f. By adding proposition symbols, the term algebra $\mathbf{T}_1$ brings about a sense of correspondence between the basic symbols used for building the set of actions and those used for building the set of formulas.

Propositions Matter

DAL, along with the variants discussed in Sec. 4.1, focuses on formalizing notions of permission and prohibition as they apply to actions. These systems, however, encounter difficulties when attempting to capture statements such as *it is not the case that you are permitted to drive without a license*. The challenge arises from the lack of a formal distinction between *pure propositions*—e.g., *you have a driver's license*—and *normative propositions*—e.g., *you are permitted to drive*. This limitation is naturally overcome in $\mathsf{DAL}(\mathsf{Prop})$, which includes propositional symbols for expressing states of affairs. In particular, in $\mathsf{DAL}(\mathsf{Prop})$, we may use $\mathsf{haslicense} \in \mathsf{Prop}$ to express that a person has a driver's license, and $\mathsf{P}(\mathsf{driving})$ to express that driving is permitted. With this distinction in place, the statement it is not the case that you are permitted to drive without a license can be precisely formalized as $\neg(\neg\mathsf{haslicense} \wedge \mathsf{P}(\mathsf{driving}))$. This enriched expressive power allows $\mathsf{DAL}(\mathsf{Prop})$ to more accurately capture the interplay between factual and normative content in deontic reasoning.

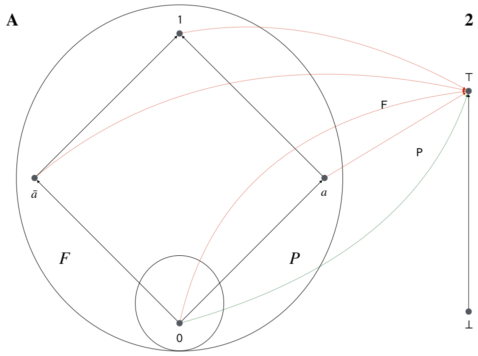

Figure 6: The Cottage Regulations Example

Including propositions in deontic action algebras opens the door to interesting and subtle forms of normative reasoning. Consider the following scenario: *there must be no fence; if there is a fence, then it must be a white fence; there is a fence*. This is a typical case of contrary-to-duty reasoning, where a secondary obligation is triggered by the violation of a primary one. As discussed in [23], such cases often involve applying normative constraints to propositions rather than actions. This shift in focus—from what ought to be done to what ought to be the case—is a hallmark of many deontic logics developed in the latter part of the 20th century, most notably Standard Deontic Logic (SDL) [6]. However, this shift is not without issues. For example, in SDL, a straightforward formalization of the scenario above, combined with the seemingly innocuous assumption that if there is a white fence, then there is a fence, results in a contradiction—despite the fact that the original scenario appears entirely coherent [10].

The position taken in [23] is that it is worthwhile to explore conditions under which contrary-to-duty statements can be given consistent and meaningful interpretations. In this spirit, we offer the following observation. So far, we have treated the algebra $\mathbf{A}$ in a deontic action algebra $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$ as representing actions. However, a more abstract perspective is also possible: the elements of $\mathbf{A}$ can be viewed more generally as *prescribable entities*—i.e., things that can be subject to normative evaluation. These may include not only actions, but also certain propositions, such as *the fence is white*. Under this broader reading, DAL(Prop) allows us to distinguish between propositions over which normative judgments can be meaningfully expressed, and those over which they cannot. For example, a statement like *it is permitted that it is raining* seems odd: the fact that it is raining is a descriptive matter, not something typically subject to norms. In this case, it is raining would be treated as a proposition in $\mathbf{F}$, outside the scope of prescribable content in $\mathbf{A}$. To better understand how this new perspective on prescribable entities works in practice, we revisit the fence example and analyze it by drawing a distinction between propositions that can be subject to normative prescriptions and those that cannot.

Example 5. *Let us use* $\mathsf{O}(\alpha)$*, read as* α *is* obligatory*, as an abbreviation of* $\mathsf{F}(\bar{\alpha})$*. Then, we use the formulas:* $\mathsf{O}(\overline{\mathsf{isfenced}})$ *to indicate that* there must be no fence*,* $\mathsf{isfenced} = 1 \to \mathsf{O}(\mathsf{isfenced} \sqcap \mathsf{iswhite})$ *to indicate that* if there is a fence, then, it must be a white fence*, and* $\mathsf{isfenced} = 1$ *to indicate that* there is a fence*. Consider now the deontic action algebra* $\mathbf{D}$ *in Fig. 6, together with the interpretation* $h : \mathbf{T}_1 \to \mathbf{D}$ *defined as* $h_{\mathsf{a}}(\mathsf{isfenced}) = 1$*,* $h_{\mathsf{a}}(\mathsf{iswhite}) = a$*. In this deontic action algebra, we have:*

1. $h(\mathsf{O}(\overline{\mathsf{isfenced}})) =_{\mathbf{F}} \top$.
2. $h(\mathsf{isfenced} = 1 \to \mathsf{O}(\mathsf{isfenced} \sqcap \mathsf{iswhite})) =_{\mathbf{F}} \top$.
3. $h(\mathsf{isfenced} = 1) =_{\mathbf{F}} \top$.

In brief, the deontic action algebra $\mathbf{D}$ *shows that drawing a distinction between prescribable and non-prescribable entities results in a consistent formalization of the fence scenario. This stands in contrast to* SDL*, where a similar formalization leads to a contradiction. By interpreting the elements of* $\mathbf{A}$ *as prescribable entities, the framework supports a more nuanced and flexible handling of contrary-to-duty reasoning. Specifically, the pure proposition* there is a fence *is formalized* $\mathsf{isfenced} = 1$*, while normative statements such as* it is obligatory that there is no fence *and* it is obligatory that there is a fence *are formalized as* $\mathsf{O}(\overline{\mathsf{isfenced}})$ *and* $\mathsf{O}(\mathsf{isfenced})$*, respectively. Importantly, although both* $\mathsf{O}(\overline{\mathsf{isfenced}})$ *and* $\mathsf{O}(\mathsf{isfenced})$ *evaluate to* $\top$ *in* $\mathbf{D}$ *under* h*, this does not result in a contradiction—unlike* SDL*.*

The discussion and examples presented above are particularly relevant to understand some broader implications of Segerberg's formalization of permission and prohibition. In particular, we showed that while deontic action algebras can be used to model systems in the ought-to-be tradition, they also enable for a principled distinction between entities that can meaningfully be regulated (those in $\mathbf{A}$) and those that cannot (those in $\mathbf{F}$).

It is worth pointing out that $\mathsf{DAL}(\mathsf{Prop})$ is not without difficulties, some of which are shared with other ought-to-do frameworks [10]. A central point of caution is the intuitive reading of certain formulas. For instance, $\mathsf{P}(\alpha) \to \mathsf{P}(\alpha \sqcap \beta)$ could naively be read as: *if it is permitted to smoke, then it is permitted to smoke and kill.* Such an interpretation is misleading unless one recalls that $\mathsf{P}(\alpha)$ expresses a notion often referred to as *strong permission*—an action α is permitted under all circumstances. A weaker notion of permission captures that an action is permitted only in certain contexts can be expressed via the negation of prohibitions, e.g., $\neg\mathsf{F}(\alpha)$. This distinction between strong and weak permission highlights both the expressive flexibility of the framework and the need for careful interpretation of the logic to normative reasoning.

4.3 Heyting Algebras for Formulas

Let us now turn to leveraging the modular framework of deontic action algebras in the construction of new deontic action logics. In Sec. 3, we brought attention to this modularity presenting a deontic action algebra as a structure $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F} \rangle$, with $\mathbf{A}$ and $\mathbf{F}$ interpreting actions and formulas, and $=$, P, and F formalizing equality, permission, and prohibition of actions. While we have primarily considered $\mathbf{A}$ and $\mathbf{F}$ as Boolean algebras, our framework allows also for alternative algebras for actions and

H1. $\varphi \to (\varphi \vee \psi)$	H7. $\bot \to \varphi$
H2. $\varphi \to (\psi \vee \varphi)$	H8. $\varphi \to \top$
H3. $\varphi \wedge \psi \to \varphi$	H9. $\varphi \to (\psi \to \varphi)$
H4. $\varphi \wedge \psi \to \psi$	H10. $\varphi \to (\psi \to (\varphi \wedge \psi))$
H5. $(\varphi \to \bot) \to \neg\varphi$	H11. $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$
H6. $\neg\varphi \to (\varphi \to \bot)$	H12. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

Figure 7: Axiom System of $\mathsf{DAL(IPL)}$

formulas. Notably, defining $\mathbf{F}$ as a Heyting algebra leads to a new deontic action logic worth considering. We call this new logic $\mathsf{DAL(IPL)}$. We begin with an outline of the technical foundations of $\mathsf{DAL(IPL)}$, and follow with a discussion of its key features and advantages.

Constructive Reasoning in Deontic Action Algebras

The language of $\mathsf{DAL(IPL)}$ contains the actions and formulas of $\mathsf{DAL(Prop)}$. Namely, actions are built using basic action symbols in Act_0, and the connectives $\sqcup$, $\sqcap$, $\bar{\ }$, $\mathsf{0}$, and $\mathsf{1}$. In turn, formulas are built using proposition symbols in Prop, the deontic connectives on actions, i.e., $\mathsf{P}(\alpha)$ and $\mathsf{F}(\alpha)$, and the connectives $\vee$, $\wedge$, $\neg$, $\bot$, and $\bot$. The sole difference is that $\mathsf{DAL(IPL)}$ introduces the connective $\to$ as primitive rather than as an abbreviation —with $\varphi \leftrightarrow \psi$ remaining as an abbreviation for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. The axiomatization of $\mathsf{DAL(IPL)}$ uses the axioms in Fig. 3 for actions, equality, and the deontic operations, while the axioms for the propositional connectives (A1'–A13' and LEM') are replaced by those in Fig. 7. These last axioms are standard for Intuitionistic Propositional Logic [24]. Provability and theoremhood are straightforwardly adapted to accommodate for the new axioms.

The algebraization of $\mathsf{DAL(IPL)}$ replaces the Boolean algebra of formulas in the definition of a deontic action algebra for a Heyting algebra. The precise definition of the new type of deontic action algebra being used is given below.

Definition 20. *A BH-deontic-action algebra is an algebra* $\mathbf{D} = \langle \mathbf{A}, \mathbf{H}, =, \mathsf{P}, \mathsf{F} \rangle$ *where:* $\mathbf{A}$ *is a Boolean algebra,* $\mathbf{H}$ *is a Heyting algebra, and* $= : |\mathbf{A}| \times |\mathbf{A}| \to |\mathbf{H}|$, $\mathsf{P} : |\mathbf{A}| \to |\mathbf{H}|$, *and* $\mathsf{F} : |\mathbf{A}| \to |\mathbf{H}|$ *satisfy the conditions 1–6 in Def. 14.*

In brief, Heyting algebras play a role in constructive reasoning analogous to the role Boolean algebras play in classical reasoning. A key distinction is how Heyting algebras treat $\to$. Despite this difference, Heyting algebras are closely related to Boolean algebras. Specifically, every Boolean algebra is a Heyting algebra, and the regular elements of a Heyting algebra —those $x \in |\mathbf{H}|$ for which $x =_{\mathbf{F}} \neg\neg x$ —form a Boolean algebra. It is well-known also that Heyting algebras have a representation theorem —as the category of Heyting algebras is dually equivalent to the category of Eusaki spaces. Furthermore, the Lindenbaum algebra obtained from the axioms in Fig. 7 is itself a Heyting algebra [25]. These facts collectively support the idea of replacing Boolean algebras with Heyting algebras in the algebraic treatment of deontic action logic, ensuring that such an approach is well-founded.

The proposition below exposes an interesting feature of BH-deontic-action algebras.

Proposition 17. *Let* $\mathbf{D} = \langle \mathbf{A}, \mathbf{H}, =, \mathsf{P}, \mathsf{F} \rangle$ *be a BH-deontic-action algebra. The pre-images* P *and* F *of* $\top$ *under* P *and* F*, respectively, are ideals in* $\mathbf{A}$ *s.t.* $P \cap F = \{\mathsf{0}\}$.

Proof. Note that ideals in Heyting algebras and ideals in Boolean algebras are defined identically. Note also that the proof of the analogous result for deontic action algebras in Prop. 5 uses only reasoning on ideals and the properties of $=$, P, and F. Since these properties are maintained in BH-deontic-action algebras, the proof in Prop. 5 transfers directly to this new setting. □

In line with Prop. 5, the result in Prop. 17 tells us that permission and prohibition on actions yielding ideals carry over if we replace Boolean for Heyting algebras.

We conclude the technical presentation of $\mathsf{DAL}(\mathsf{IPL})$ with soundness and completeness theorems for the logic. Definitions of interpretations of the term algebra into BH-deontic-action algebras, homomorphisms, and congruences and quotients, are akin to those in Sec. 3.

Theorem 18. *Let* $\mathbb{BH}$ *be the class of all BH-deontic-action algebras. A formula* φ *is a theorem of* $\mathsf{DAL}(\mathsf{IPL})$ *iff* $\mathbb{BH} \models \varphi \doteq \top$.

Proof. Like with the proofs of Thms. 15 and 16, we only remark on the differences with the proofs of Thms. 6 and 9.

Soundness. We need to prove that any interpretation of an axiom is mapped to $\top$. For axioms on actions, this is just like in Thm. 6. For axioms on propositional connectives, this is immediate from well known results for Heyting algebras (see [24, 25]). Finally, the cases of equality, permission, and prohibition are not affected by the new interpretation of $\to$ in a Heyting algebra.

Completeness. We begin by defining the Lindenbaum algebra as in Prop. 8 via congruences $\cong_{\mathsf{a}}$ for actions and $\cong_{\mathsf{f}}$ for formulas. Again, following [25], it is easy to see that the axioms in Fig. 7 result in the algebra of formulas itself being a Heyting algebra. This construction provides a witness for theoremhood for the logic. □

Constructive Reasoning Matters

As with our earlier discussion of propositions, let us motivate why interpreting formulas over Heyting algebras—rather than Boolean algebras—has an interest that goes beyond purely formal considerations.

To set the stage, imagine the following scenario: *if John does not have a driver's license, then, it is forbidden for him to drive; John is not forbidden to drive.* From this scenario, using classical reasoning, we would infer that *John has a driver's license.* This conclusion is somewhat counterintuitive. We could argue that the absence of a prohibition to drive (for some particular reason that we do not know) does not directly ensures that John has applied and obtained a driving license. Perhaps, under some exceptional circumstances—John must drive to save a life—he is permitted to drive even without a license.

Counterintuitive inferences such as the one above suggests a different approach: instead of reasoning classically, we can adopt an intuitionistic perspective. This shift

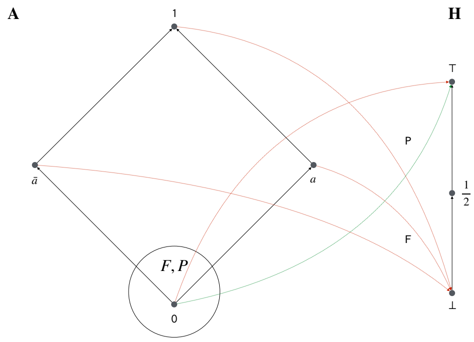

Figure 8: The Driver's License Paradox.

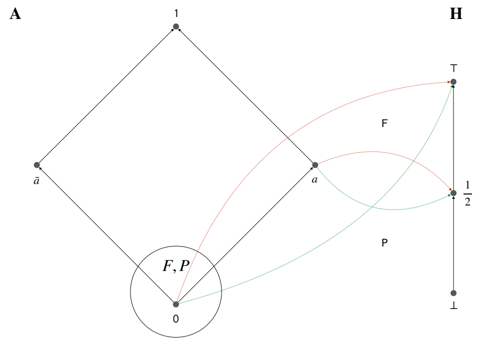

Figure 9: Principle of Deontic Closure.

prevents unwarranted conclusions in scenarios like John's and offers a more nuanced way of modeling such situations.

Example 6. *The driver's license example in the previous paragraph can be formalized with formulas:* $\neg\mathsf{haslicense} \to \mathsf{F}(\mathsf{driving})$ *capturing that* if John does not have a driver's license, then, it is forbidden for him to drive*, and* $\neg\mathsf{F}(\mathsf{driving})$ *capturing that* John is not forbidden to drive*. Note that, in this formalization,* $\mathsf{haslicense} \in \mathsf{Prop}$ *and* $\mathsf{driving} \in \mathsf{Act}_0$*. The BH-deontic-action algebra* $\mathbf{D}$ *in Fig. 8, together with the interpretation* h *defined as* $h_{\mathsf{a}}(\mathsf{driving}) = a$*,* $h_{\mathsf{f}}(\mathsf{haslicense}) = \frac{1}{2}$*, prove that* $\mathsf{haslicense}$ *is not a consequence of the previous two formulas. Precisely, we have:*

1. $h(\neg\mathsf{haslicense}) =_{\mathbf{H}} \neg h(\mathsf{haslicense}) =_{\mathbf{H}} \neg\frac{1}{2} =_{\mathbf{H}} \bot$.
2. $h(\neg\mathsf{haslicense} \to \mathsf{F}(\mathsf{driving})) =_{\mathbf{H}} \bot \to h(\mathsf{F}(\mathsf{driving})) =_{\mathbf{H}} \top$.
3. $h(\neg\mathsf{F}(\mathsf{driving})) =_{\mathbf{H}} \neg h(\mathsf{F}(\mathsf{driving})) =_{\mathbf{H}} \neg\bot =_{\mathbf{H}} \top$.
4. $h(\mathsf{haslicense}) \neq_{\mathbf{H}} \top$.

Note how in the BH-deontic-algebra $\mathbf{D}$ *in Fig. 8 the only element of the algebra of actions which the operation* F *maps to* $\top$ *is* 0*, all other elements are mapped to* $\bot$*.*

There is another interesting discussion emerging from the use of an intuitionistic basis for reasoning about formulas. Recall the deontic action logic $\mathsf{DAL}(1)$ from Sec. 4.1 and how this logic is built from DAL adding additional axioms with the intent to capture the *principle of deontic closure*. This principle is stated in [8] as: *what is not forbidden is permitted*. The formalization of this principle as formulas of the form $\mathsf{F}(\mathsf{a}) \vee \mathsf{P}(\mathsf{a})$ is taken from [14]. Still, a more faithful formalization of this principle is $\neg\mathsf{F}(\mathsf{a}) \to \mathsf{P}(\mathsf{a})$. Clearly, there is no substantial distinction in a classical setting, as both formulas are equivalent. This is not the case in an intuitionistic setting. For instance, the BH-deontic-action algebra $\mathbf{D}$ in Fig. 9 satisfies one version of the principle but not the other. Precisely, note that if we have a single basic action symbol $\mathsf{a} \in \mathsf{Act}_0$, and an interpretation h on $\mathbf{D}$ s.t. $h_{\mathsf{a}}(\mathsf{a}) = a$, then:

1. $h(\mathsf{F}(\mathsf{a})) =_{\mathbf{H}} \mathsf{F}(h(\mathsf{a})) =_{\mathbf{H}} \mathsf{F}(a) =_{\mathbf{H}} \frac{1}{2}$.
2. $h(\neg\mathsf{F}(\mathsf{a})) =_{\mathbf{H}} \neg h(\mathsf{F}(\mathsf{a})) =_{\mathbf{H}} \neg\frac{1}{2} =_{\mathbf{H}} \bot$.
3. $h(\mathsf{P}(\mathsf{a})) =_{\mathbf{H}} \mathsf{P}(h(\mathsf{a})) =_{\mathbf{H}} \mathsf{P}(a) =_{\mathbf{H}} \frac{1}{2}$.
4. $h(\neg\mathsf{F}(\mathsf{a}) \to \mathsf{P}(\mathsf{a})) =_{\mathbf{H}} \bot \to h(\mathsf{P}(\mathsf{a})) =_{\mathbf{H}} \top$.
5. $h(\mathsf{F}(\mathsf{a}) \vee \mathsf{P}(\mathsf{a})) =_{\mathbf{H}} \frac{1}{2} \vee \frac{1}{2} =_{\mathbf{H}} \frac{1}{2} \neq_{\mathbf{H}} \top$.

In words, this example shows that there is a distinction between considering the principle of deontic closure as *what is not forbidden is permitted* —alternatively, *what is not permitted is forbidden*—and considering this principle as *every (basic) action is either permitted or forbidden.*

To sum up, we have explored some key features and applications of replacing the Boolean algebra of formulas in a deontic action algebra for a Heyting algebra. The results we obtained underscore leveraging the modularity of our framework to build a new deontic action logic $\mathsf{DAL}(\mathsf{IPL})$. The discussion and ensuing examples reinforce the utility of this new logic in the broader area of Deontic Logic, and in particular in relation to the principle of deontic closure.

4.4 A Heyting Algebra of Actions

The formal machinery in Sec. 4.3 naturally suggests its symmetric extension: replacing the Boolean algebra of actions with a Heyting algebra. This results in a new deontic action logic, $\mathsf{DAL}(\mathsf{IAL})$, with actions interpreted in an intuitionistic framework. To the best of our knowledge, no existing deontic action logic provides an intuitionistic perspective on actions, making this approach a novel contribution.

Another look at Constructive Reasoning in Deontic Action Algebras

We begin with an outline of the technical foundations of $\mathsf{DAL}(\mathsf{IAL})$. The formulas of this new logic are built using proposition symbols in Prop, the deontic connectives on actions, i.e., $\mathsf{P}(\alpha)$ and $\mathsf{F}(\alpha)$, and the connectives $\vee$, $\wedge$, $\neg$, $\bot$, and $\bot$. In turn, actions are built using basic action symbols in Act_0, and the connectives $\sqcup$, $\sqcap$, $\bar{\ }$, $\mathsf{0}$, and $\mathsf{1}$. In addition, $\mathsf{DAL}(\mathsf{IAL})$ introduces a new connective $\dashv$ on actions giving rise to actions of the form $\alpha \dashv \beta$. This new connective is introduced to capture the notion of a relative complement (or intuitionistic implication) in a Heyting algebra. The axiomatization of $\mathsf{DAL}(\mathsf{IAL})$ uses all the axioms in Fig. 3 except the axiom (LEM) for actions. In addition, it introduces as axioms the properties H1–H3 in Def. 10 for the new connective $\dashv$. In essence, the axioms for actions are the conditions on Heyting algebras in [26]. Provability and theoremhood are easily adapted to accommodate for the new axioms.

The algebraization of $\mathsf{DAL}(\mathsf{IAL})$ replaces the Boolean algebra of actions in the definition of a deontic action algebra for a Heyting algebra. This is made precise in Def. 21 below.

Definition 21. *An HB-deontic-action algebra is an algebra* $\mathbf{D} = \langle \mathbf{H}, \mathbf{B}, =, \mathsf{P}, \mathsf{F} \rangle$ *where:* $\mathbf{H}$ *is a Heyting algebra,* $\mathbf{B}$ *is a Boolean algebra, and* $= : |\mathbf{H}| \times |\mathbf{H}| \to |\mathbf{B}|$, $\mathsf{P} : |\mathbf{H}| \to |\mathbf{B}|$*, and* $\mathsf{F} : |\mathbf{H}| \to |\mathbf{B}|$ *satisfy the conditions 1–6 in Def. 14.*

Prop. 19 shows that permission and prohibition behave as expected.

Proposition 19. *Let* $\mathbf{D} = \langle \mathbf{H}, \mathbf{B}, =, \mathsf{P}, \mathsf{F} \rangle$ *be an HB-deontic-action algebra. The pre-images* P *and* F *of* $\top$ *under* P *and* F*, respectively, are ideals in* $\mathbf{A}$ *s.t.* $P \cap F = \{\mathsf{0}\}$*.*

Proof. Analogous to that in Prop. 17. □

Since $\mathsf{DAL}(\mathsf{IAL})$ is the symmetric counterpart of $\mathsf{DAL}(\mathsf{IPL})$, the proofs of soundness and completeness for $\mathsf{DAL}(\mathsf{IAL})$ can be straightforwardly adapted from those of $\mathsf{DAL}(\mathsf{IPL})$. Consequently, we establish the following theorem.

Theorem 20. *Let* $\mathbb{HB}$ *be the class of all HB-deontic-action algebras. It follows that* φ *is a theorem of* $\mathsf{DAL}(\mathsf{IAL})$ *iff* $\mathbb{HB} \models \varphi \doteq \top$*.*

Constructive Reasoning and Realization of Actions

We put forth the argument that an intuitionistic basis for actions is particularly valuable when actions are understood as constructions that witness their realizability. This perspective mirrors the semantics of Intuitionistic Logic (IL), where the truth of a formula is tied to the existence of a proof.

An intuitionistic interpretation of actions is not only theoretically interesting, it may also be practically motivated. For instance, in the context of automated planning [27], consider a robot tasked with executing various actions. To perform an action, the robot must possess a *plan*—a concrete sequence of steps that realizes the action. In this setting, classical assumptions such as $\bar{a} \sqcup a = \mathsf{1}$ may no longer hold; since the robot might have no plan for a, nor any means to determine if a is altogether unrealizable.

This new interpretation of actions naturally extends to operations on actions. Especially interesting are the cases of implication ($\dashrightarrow$) and complement ($^{-}$), which can be understood as transformations on plans—mirroring how implication and negation are treated in IL. Specifically, the action $a \dashrightarrow b$ represents a functional transformation: it provides a method for converting any plan that realizes a into a plan that realizes b—just as intuitionistic implication transforms any proof of the antecedent into a proof of the consequent. Similarly, $\bar{a}$, defined as $a \dashrightarrow 0$, can be seen as a transformation that maps any plan for a into a plan for an impossible action. In this way, instead of denoting the performance of some alternative action, $\bar{a}$ expresses the unrealizability of a, capturing a failure of constructibility—analogous to intuitionistic negation, where a negated formula shows that any proof of it would lead to a contradiction.

It is clear that prescriptions often play an important role in planning. For instance, if the robot is an autonomous vehicle, it must adhere to transit rules. In this case, $\mathsf{P}(\alpha)$ indicates that plans for executing α are permitted, while $\mathsf{F}(\alpha)$ signals that such plans are forbidden.

In sum, adopting an intuitionistic perspective on actions yields a coherent and conceptually robust account of actions as constructive processes. This interpretation naturally extends to implication and complement of actions as transformations on plans, mirroring the foundational principles of constructive logic. Beyond its theoretical appeal, an intuitionistic interpretation of actions proves to be practically valuable in contexts where executing an action depends on constructing a realizable plan, like automated planning. In this light, $\mathsf{DAL}(\mathsf{IAL})$ emerges as a logical framework that not only captures the intuitionistic structure of action-based reasoning, but also accommodates practical considerations involving realizability and normative constraints.

4.5 Intuitionistic Deontic Action Logic

Clearly, we can also simultaneously replace the Boolean algebras for actions and formulas for Heyting algebras. We call the resulting logic $\mathsf{DAL}(\mathsf{INT})$. Similarly to the case in Sec. 4.4, to the best of our knowledge, this logic is the first fully intuitionistic deontic action logic.

The Logic Itself

The language and axiomatization of $\mathsf{DAL}(\mathsf{INT})$ combines those of $\mathsf{DAL}(\mathsf{IAL})$ and $\mathsf{DAL}(\mathsf{IPL})$ in Secs. 4.3 and 4.4. Precisely, it builds actions like in $\mathsf{DAL}(\mathsf{IAL})$—using $\multimap$ as a primitive connective. In turn, it builds formulas like in $\mathsf{DAL}(\mathsf{IPL})$—using $\to$ as a primitive connective. The axiomatization of this new logic takes the axioms for actions from $\mathsf{DAL}(\mathsf{IAL})$ and the axioms for formulas from $\mathsf{DAL}(\mathsf{IPL})$. The logic retains the axiomatization of equality, permission, and prohibition of Segerberg's logic, i.e., axioms E1–E2 and D1–D3 in Fig. 3. The notions of proof and theoremhood are reformulated in the obvious way.

The algebraization of $\mathsf{DAL}(\mathsf{INT})$ replaces the Boolean algebras of actions and of formulas in the definition of a deontic action algebra for Heyting algebras. This is made precise in the definition of an *intuitionistic deontic action algebra* below.

Definition 22. *By an intuitionistic deontic action algebra, we mean an algebra* $\mathbf{D} = \langle \mathbf{A}, \mathbf{H}, =, \mathsf{P}, \mathsf{F}\rangle$ *where:* $\mathbf{A}$ *and* $\mathbf{F}$ *are Heyting algebras, and* $= : |\mathbf{A}| \times |\mathbf{A}| \to |\mathbf{F}|$, $\mathsf{P} : |\mathbf{A}| \to |\mathbf{F}|$, *and* $\mathsf{F} : |\mathbf{A}| \to |\mathbf{F}|$ *satisfy the conditions 1–6 in Def. 14.*

As before, permission and prohibition behave as expected.

Proposition 21. *Let* $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, =, \mathsf{P}, \mathsf{F}\rangle$ *be an intuitionistic deontic action algebra. The pre-images* P *and* F *of* $\top$ *under* P *and* F *are ideals in* $\mathbf{A}$ *s.t.* $P \cap F = \{\mathsf{0}\}$.

Proof. We can reuse the proof of Prop. 17 as it only uses the properties of P and F plus absorption, and idempotence properties which also hold in Heyting algebras. $\square$

For stating and proving the soundness and completeness of $\mathsf{DAL}(\mathsf{INT})$, we define interpretations and algebraic validity as in previous sections.

Theorem 22. *Let* $\mathbb{ID}$ *be the class of all intuitionistic deontic action algebras. Then,* φ *is a theorem of* $\mathsf{DAL}(\mathsf{INT})$ *iff* $\mathbb{ID} \models \varphi \doteq \top$.

Proof. We obtain this result by putting together the intuitionistic parts of the proofs of Thms. 18 and 20. $\square$

Intuitionistic Deontic Action Logic in Practice

The logic $\mathsf{DAL}(\mathsf{INT})$ may be useful for reasoning in scenarios where there is partial observability about the state of affairs, typical in reinforcement learning and planning. Consider the following example adapted from [28]: a robot is tasked with cleaning an office and needs to reach certain spots. The robot has sensors to detect doorways, walls, or open spaces, but the sensor information may sometimes be unclear. Proposition symbols like north, south, east, and west could represent the robot's orientation, while $\mathsf{doorway}$, wall, and clear capture the information provided by the sensors. Uncertainty entails the robot might fail to determine whether there is a doorway ahead or not, i.e., $\mathsf{doorway} \vee \neg\mathsf{doorway}$ may fail to hold, violating the law of the excluded middle at the level of formulas. The example can be extended with an intuitionistic model of actions. For instance, we can consider actions $\mathsf{advance}$ and rotate for the robot moving forward and rotating, respectively. As in the example in Sec. 4.4, the interpretation of these actions is tied to a plan allowing the robot to realize the action. Once again, a formula $\mathsf{advance} \sqcup \overline{\mathsf{advance}} = 1$ may fail to hold (violating the law of excluded middle at the level of actions) because the robot lacks a plan due to incomplete sensor information.

Finally, we can spice up this scenario with prescriptions. For instance, there might be signals in the corridors indicating the robot is not to cross certain doorway, prohibiting such an action. Again, the robot may lack sufficient information to determine whether $\mathsf{F}(\mathsf{advance}) \vee \neg\mathsf{F}(\mathsf{advance})$ holds or not.

In all the above scenarios, having an intuitionistic deontic action logic like $\mathsf{DAL}(\mathsf{INT})$ provides a formal framework for analysis in which we can address issues arising from partial observability and prescriptive constraints.

5 Final Remarks

We developed an algebraic framework for Deontic Action Logic (DAL) and its variations using deontic action algebras. These structures consist of two Boolean algebras connected by operations that capture key aspects of permission and prohibition. We showed that the algebraic characterization is adequate by proving soundness and completeness theorems. We discussed the advantages of our algebraic approach for modelling scenarios through concrete examples. Our algebraic treatment of DAL can be thought of as an abstract version of deontic action logics which can be used to establish connections between deontic action logics and areas such as topology, category theory, probability, etc. Moreover, the framework is modular. In Sec. 4, we showed how it is possible to select the adequate underlying algebraic structures to develop new logics, highlighting the flexibility and extensibility of our approach.

We introduced deontic action algebras in [17]. In this article, we extended our previous work and explored the inclusion of Heyting algebras to obtain intuitionistic behavior. Our approach accommodates such a formulation in a very simple manner, paving the way for interesting future work. Several alternative algebraic structures for actions and propositions warrant further investigation. In particular, we aim to characterize action composition and action iteration. These operations are not foreign in deontic reasoning. The work in [29] on Dynamic Deontic Logic (DDL) was one of the first in considering a deontic logic containing action composition. This treatment, however, is not without challenges [30]. Regarding action iteration, Broersen observed in [12] that dynamic deontic logics vary in how deontic operators apply to composed actions. Specifically, the distinction is between: (i) *goal norms*, where the deontic evaluation of a sequence of actions depends on the outcome of the final action (e.g., the sequence is permitted or forbidden based on whether the final state it reaches is permitted or forbidden); and (ii) *process norms*, where the deontic evaluation of a composed sequence of actions depends on the individual actions in the sequence (e.g., a sequence is permitted or forbidden only if each—or in some variants, at least one—action in the sequence is permitted or forbidden). To the best of our knowledge, only a few extensions of DAL have incorporated action composition and iteration. A notable example is the framework presented in [**?**], where deontic action logic is enriched with sequential composition. Therein, the authors provide a semantic interpretation of sequential composition, study its interaction with deontic operators, and propose an axiomatization of the resulting logic. However, completeness of the logic is not addressed. This illustrates both the potential and the challenges of extending DAL with richer forms of action structure.

We believe that the framework of deontic action algebras is particularly well-suited for pursuing such extensions, since it can be straightforwardly adapted to accommodate richer action structures such as composition and iteration. Concretely, we may consider deontic action algebras $\langle \mathbf{A}, \mathbf{F}, \mathsf{P}, \mathsf{F}, = \rangle$ where the algebra $\mathbf{A} = \langle A, +, ;, * \rangle$ of actions is a Kleene algebra, with ; being the operation of composition, and $*$ the operation of iteration (see, e.g., [31]). Kleene algebras enjoy some nice properties, e.g., they are quasi-varieties, and they are complete w.r.t. equality of regular expressions (see [32]). These properties make them a natural candidate for extending the expressive power of DAL.

Similarly, one can extend deontic action algebras with other interesting algebras, e.g., relation algebras (see [33]) that most notably provide action converse. We leave it as further work to investigate the properties of the operators P and F in these new algebraic settings.

Beyond Boolean and Heyting algebras, it is also interesting to explore alternative algebras for propositions. Some immediate examples include: BDL-algebras, semi-lattices, and metric spaces. This may lead to the design of deontic logics that are not logics of normative propositions but logics of norms instead—a distinction that was already noted by von Wright in [34] and Alchourrón in [35, 36]. Both SDL and DAL are logics of normative propositions insofar as they assign truth values to formulas in the logic. In contrast, logics of norms can express prescriptions that do not carry with them truth values. To accommodate such logics, we can generalize the interpretation of formulas to other algebraic structures. For instance, adopting a meet semi-lattice as the algebra of propositions allows for norms to be combined while also accounting for potential contradictions among them without necessarily requiring norms to be true or false. Of course, several other algebraic frameworks could serve this purpose as well.

The versatility observed in Sec. 4 suggests a possible connection between DAL and combining logics [37]. Building on the algebraic treatment of DAL, we can derive a characterization in category-theoretic terms [38], which naturally connects to the framework of institutions introduced by Goguen and Burstall [39, 40]. Institutions provide an abstract framework for model theory that captures the essence of logical systems and their combinations, offering a unified perspective on how logics can be integrated through categorial constructions. Many of these methods are unified within the algebraic fibring approach introduced by Sernadas, Sernadas, and Caleiro [41], which significantly enhances the versatility of logic combination through universal categorial constructions. This approach extends the range of logics that can be combined beyond modal logics, demonstrating a fruitful interplay between the algebraic and categorial perspectives. Altogether, these frameworks establish a strong link between DAL and broader methodologies for combining logics, reinforcing the relevance of algebraic and categorial approaches to logical systems.

Finally, the algebraization of DAL in this article provides a rich foundation for studying deontic action logics dynamically, *á la* Public Announcement Logic [42]. In particular, we note that the algebraic semantics of deontic operators induces a restriction on the algebra of formulas. Such a restriction resembles the model update operators in dynamic logics, suggesting an interesting parallel and potential applications in modeling evolving normative systems. Exploring these connections, alongside the broader algebraic and categorial perspectives outlined above, offers a promising direction for future research.

Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Data Availability Statement

No new data were created or analyzed in this paper.

Acknowledgements. This work is supported by Agencia I+D+i grants PICT-2020-3780 and PICT-2021-00400, Stic-AmSud 23-STIC-07 'DL(R)', Secyt-UNC grant 33620230100178CB, the EU Grant Agreement 101008233 (MISSION), the IRP SINFIN, and the France 2030 program ANR-11-IDEX-0003.

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