The Dynamics of Knowing How

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Abstract

We investigate dynamic operations acting over a knowing how logic. Our approach uses a recently introduced semantics for the knowing how operator, which is based on an indistinguishability relation between plans, and which is arguably closer to the standard presentation of knowing that modalities in classic epistemic logic. Here, we discuss how the new indistinguishability-based semantics enables us to define dynamic modalities representing different ways in which an agent can learn how to achieve a goal. In this regard, we study different alternatives for implementing two types of updates: ontic and epistemic. For the former, we provide axiomatizations over a restricted class of models. For the latter, we investigate some semantic properties and discuss what the difficulties are in axiomatizing dynamic modalities. In turn, we introduce a novel dynamic epistemic modality for which we have reduction axioms over an extended static knowing how language.

Keywords: Knowing how, updates, axiomatization, decidability

1 Introduction

The notion of *knowing that* has been one of the fundamental study subjects in Epistemology. It concerns the knowledge an agent might have about the truth-value of propositions, and as such has driven most of the field's research agenda, with some authors even claiming that it subsumes many other forms of knowledge (see, e.g., [1–3]).

One of the most successful tools for studying this concept has been Epistemic Logic (EL; [4]), a formal logical framework which, on the semantic side, typically relies on pointed Kripke models [5, 6]. In more detail, a Kripke model is a relational structure whose domain's elements (called possible states/situations/worlds) represent the different ways the real world can be, and in which a binary *epistemic indistinguishability* relation represents the agent's uncertainty. Then, a pointed Kripke model is a Kripke model with a distinguished evaluation state, usually understood as describing the actual situation. With this, knowledge about the truth-value of propositions is defined as follows: at a given state w, the agent knows that a proposition φ is true if and only if φ holds in all the states she cannot distinguish from w. Asking for the indistinguishability relation to be reflexive, transitive and Euclidean (i.e., an equivalence relation) makes knowledge truthful, positively and negatively introspective. EL has proved to be a useful tool for developments in, e.g., Philosophy [7, 8], Computer Science [9], AI [10] and Economics [11]. In achieving this, one of its most appealing aspects is that it can be used to reason about information change. Indeed, actions that increase the agent's knowledge can be straightforwardly represented as model operations that remove edges, thus reducing the agent's uncertainty. Further research on this idea has given rise to Dynamic Epistemic Logic (DEL), a field that studies how the attitudes of a set of agents change through diverse informational actions (see, e.g., [12, 13]).

Still, knowing that is not the only form of knowledge an agent might have/use. In fact, the last few years have witnessed the emergence of formal frameworks for reasoning about knowing whether, knowing how, knowing why, knowing what, and so on (see [14] for an overview). Among them, knowing how (e.g., [15]) is one of the most important, as it concerns the *ability* an agent has to achieve a given outcome. Intuitively, an agent knows how to achieve φ given ψ if she has at her disposal a suitable *course of action* guaranteeing that φ will be the case whenever she is in a situation in which ψ holds. It is worthwhile to study this notion not only from a philosophical perspective, but also from a computer science point of view, as the concept can be seen, for example, as a formal account for automated planning and strategic reasoning in AI (see, e.g., [16]).

Most traditional approaches for representing knowing how rely on combining logics of knowing that with logics of action (see, e.g., [17–19]). However, while a combination of operators for knowing that and *ability* (e.g., [20]) produces a *de dicto* reading of the concept (*"the agent knows she has an action that guarantees the goal"*), a proper notion of *"knowing how to achieve φ"* requires a *de re* clause (*"the agent has an action that she knows guarantees the goal"*; see [21, 22] for a discussion). Based on these considerations, [23, 24] introduced a framework based on a knowing how binary modality $\mathsf{Kh}(\psi, \varphi)$. At the semantic level, this language is also interpreted over relational models — called labeled transition systems (LTSs) in this context. The difference with respect to the EL setting is that, here, relations describe the actions an agent considers she has at her disposal (in some sense, her *epistemic abilities*). Then, $\mathsf{Kh}(\psi, \varphi)$ holds if and only if there is a "proper plan" (a sequence of actions satisfying certain constraints) in the LTS that unerringly leads from every ψ-state only to φ-states.

While variants of this idea have been explored in the literature (see, e.g., [25–29]), most of them share the fundamental feature mentioned above: relations in the model are interpreted as the agent's epistemic take on her available actions. Due to this fact, the epistemic abilities of an agent depend *only* on what these actions can achieve. The framework presented in [30, 31] changed this by adding a notion of indistinguishability between plans, related to the notion of *strategy indistinguishability* (e.g., [32, 33]). Two main insights are introduced in these articles. The first is simple: while certain plans might be available in a given environment, they might not be available *to a given agent* (e.g., she might not know about them, or did not consider them at the given time). Hence, not all possible plans are available to all agents. More importantly, she might consider some of these plans *indistinguishable* from some others.[1] In such cases, having available a proper plan σ that leads from any ψ-state only to φ-states is not enough: the agent also needs for *all the plans she cannot distinguish from* σ to satisfy the same requirement. As argued in [30], the benefits of these new semantics are threefold. First, it provides an indistinguishability-based view of an agent's epistemic abilities, akin to the EL approach for modeling knowing that. Second, it can deal with multi-agent scenarios more naturally. Third, it leads to a natural definition of operators representing dynamic aspects of knowing how, akin to the DEL approach for modeling dynamics of knowing that.

This article focuses on the latter point, using this indistinguishability-based semantics to study several dynamic operators describing changes in the agents' epistemic abilities. To the best of our knowledge, this is the first time this theme is addressed (except by the brief discussion in [24] about knowing how and announcements, and the work in [34] that this article extends). The proposals follow the DEL approach, using changes in the model to represent changes in the situation they describe. In doing so, we take advantage of the fact that, in these structures, there is a clear distinction between ontic and epistemic information. Indeed, while the underlying LTS contains *ontic* facts common to all agents (the available actions as well as their effects), the indistinguishability relation over plans represents the *epistemic* perception of each particular agent (the plans an agent considers available, as well as her capacity to distinguish between them). Thus, model-update operations affecting different parts of the model have a clear-cut interpretation: while changes in the LTS can be seen as ontic updates (possibly with epistemic consequences), changes in plan indistinguishability relations can be seen as direct epistemic ones.

The rest of the article is organized as follows. Sec. 2 recalls the basic definitions of the indistinguishability-based knowing how setting [30, 31], including some new results. In particular, we show that one of the axioms presented in the axiomatization introduced in [30, 31] is derivable from the others, while a new axiom is needed to fix a small gap in the completeness argument. We also discuss some results concerning the expressive power of the basic knowing how logic over particular classes of models.

[1] The exact meaning of "indistinguishable" is left open. In particular, it does not necessarily mean that the agent cannot tell the plans apart, it might just be that she considers the differences irrelevant.

Secs. 3 to 5 constitute the core of the contribution. Indeed, Sec. 3, investigates *ontic* updates by introducing model operations that affect the ontic components of the models: first, an operation that removes states (in the spirit of [35]), and then one that removes edges (in the spirit of [36]). Then, Secs. 4 and 5 are devoted to *epistemic* updates. Sec. 4 discusses two operations for eliminating uncertainty between plans (the first works over two given plans; the second quantifies over the parameters of the first). The dynamic operators associated with these updates were first introduced in [34]. Here we provide a more detailed analysis and complete proofs. Then, Sec. 5 introduces an epistemic update that makes a given plan distinguishable from any other, and investigates its properties. Sec. 6 closes the article by offering some final remarks and discussing future lines of work.

Among the results presented in Secs. 3 to 5, one can highlight the following. First, it is shown that the modalities capturing the model updates from Secs. 3 and 4 increase the expressivity of the basic knowing how language. Hence, an axiomatization via reduction axioms is not possible. For the ontic updates, reduction axioms can be defined when we restrict the logic to a particular class of models, but this approach does not work for the epistemic modalities. The modality associated with the update of Sec. 5 also increases the expressive power, and hence no reduction axioms exist. However, reduction axioms can be obtained by further extending the language with standard modalities for each basic action.

2 Preliminaries

This section recalls the syntax and semantics of the knowing how framework as presented in [30] as well as a complete axiomatization and a suitable notion of bisimulation (both from [31]). It also discusses some complexity and expressivity results. Throughout the text, let Prop be a countable set of propositional symbols, Act a denumerable set of action symbols, and Agt a non-empty finite set of agents.

Definition 1. Formulas of the language $\mathsf{L}_{\mathsf{Kh}_i}$ are defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi),$$

with $p \in \mathsf{Prop}$ and $i \in \mathsf{Agt}$. Other Boolean connectives are defined as usual. Formulas as $\mathsf{Kh}_i(\psi, \varphi)$ state that *"when ψ is the case, agent i knows how to make φ true"*. Define also the abbreviations $\mathsf{A}\varphi := \mathsf{Kh}_i(\neg\varphi, \bot)$ for an arbitrary i, and $\mathsf{E}\varphi := \neg\mathsf{A}\neg\varphi$.

In [23, 24], formulas are interpreted over *labeled transition systems* (LTSs): relational models in which each (basic) relation indicates the source and target of a particular type of action the agent can perform. In the setting introduced in [30], LTSs are extended with a notion of *uncertainty* between plans.

Definition 2 (Actions and plans)**.** Let Act^* be the set of finite sequences over Act. Elements of Act^* are called *plans*, with ϵ being the *empty plan*. Given $\sigma \in \mathsf{Act}^*$, we use $|\sigma|$ to denote its length (in particular, $|\epsilon| = 0$). For $0 \leq k \leq |\sigma|$, the plan σ_k is σ's initial segment up to (and including) the kth position (with $\sigma_0 := \epsilon$). For $0 < k \leq |\sigma|$, the action $\sigma[k]$ is the one in σ's kth position.

Definition 3 (Uncertainty-based LTS)**.** An *uncertainty-based* LTS *(*$\mathrm{LTS}^{\mathrm{U}}$*)* for Prop, Act and Agt is a tuple $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ where W is a non-empty set of states (the domain, also denoted by $\mathrm{D}_{\mathcal{M}}$), $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$ is a collection of binary relations on W, $\mathrm{U} = \{\mathrm{U}(i) \subseteq 2^{\mathsf{Act}^*} \setminus \{\emptyset\} \mid i \in \mathsf{Agt}\}$ assigns to every agent a non-empty collection of pairwise disjoint non-empty sets of plans (i.e., $\mathrm{U}(i) \neq \emptyset$, $\pi_1, \pi_2 \in \mathrm{U}(i)$ with $\pi_1 \neq \pi_2$ implies $\pi_1 \cap \pi_2 = \emptyset$, and $\emptyset \notin \mathrm{U}(i)$) and $\mathrm{V} : \mathrm{W} \to 2^{\mathsf{Prop}}$ is the valuation function. The tuple $\langle \mathrm{W}, \mathrm{R}, \mathrm{V} \rangle$ is called an LTS. Given an $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}$ and $w \in \mathrm{D}_{\mathcal{M}}$, the pair $(\mathcal{M}, w)$ (parentheses usually dropped) is called a *pointed* $\mathrm{LTS}^{\mathrm{U}}$.

Intuitively, $\mathrm{P}_i = \bigcup_{\pi \in \mathrm{U}(i)} \pi$ is the set of plans agent i has at her disposal (alternatively, is aware of), and each $\pi \in \mathrm{U}(i)$ is an indistinguishability class. As discussed in [30], there is a one-to-one correspondence between each $\mathrm{U}(i)$ and an indistinguishability relation $\sim_i \subseteq \mathrm{P}_i \times \mathrm{P}_i$ describing the agent's *uncertainty* over her available plans ($\sigma_1 \sim_i \sigma_2$ iff there is $\pi \in \mathrm{U}(i)$ such that $\{\sigma_1, \sigma_2\} \subseteq \pi$). The presentation used here simplifies some of the definitions that will follow.

Given her uncertainty over (a subset of) Act^*, the epistemic abilities of an agent i depend not on what a single plan can achieve, but rather on what a set of them can guarantee.

Definition 4. Given $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$ and $\sigma \in \mathsf{Act}^*$, define $\mathrm{R}_\sigma \subseteq \mathrm{W} \times \mathrm{W}$ inductively as: $\mathrm{R}_\epsilon := \{(w, w) \mid w \in W\}$ and $\mathrm{R}_{\sigma a} := \mathrm{R}_\sigma \circ \mathrm{R}_a$ (first R_σ and then R_a). Then, for $\pi \subseteq \mathsf{Act}^*$ and $U \cup \{u\} \subseteq \mathrm{W}$, define $\mathrm{R}_\pi := \bigcup_{\sigma \in \pi} \mathrm{R}_\sigma$, $\mathrm{R}_\pi(u) := \bigcup_{\sigma \in \pi} \mathrm{R}_\sigma(u)$, and $\mathrm{R}_\pi(U) := \bigcup_{u \in U} \mathrm{R}_\pi(u)$.

In what follows, we introduce the notion of strong executability of plans (see, e.g., [23, 31] for further discussions), a condition which determines that a given plan (or a set of them) is appropriate for achieving a certain goal.

Definition 5 (Strong executability of plans)**.** Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$, with $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$. A *plan* $\sigma \in \mathsf{Act}^*$ is *strongly executable* (SE) at $u \in \mathrm{W}$ if and only if $v \in \mathrm{R}_{\sigma_k}(u)$ implies $\mathrm{R}_{\sigma[k+1]}(v) \neq \emptyset$ for every $k \in [0 \ldots |\sigma| - 1]$. The set $\mathrm{SE}^{\mathcal{M}}(\sigma) := \{w \in \mathrm{W} \mid \sigma \text{ is SE at } w\}$ contains the states in $\mathcal{M}$ where σ is SE. Then, a *set of plans* $\pi \subseteq \mathsf{Act}^*$ is *strongly executable* at $u \in \mathrm{W}$ if and only if *every* plan $\sigma \in \pi$ is *strongly executable* at u. The set $\mathrm{SE}^{\mathcal{M}}(\pi) = \bigcap_{\sigma \in \pi} \mathrm{SE}^{\mathcal{M}}(\sigma)$ contains the states in $\mathcal{M}$ where π is SE.

Thus, while *a plan* is strongly executable (at a state) when *all its partial executions* (from that state) can be completed, *a set of plans* is strongly executable when *all its plans* are strongly executable. When the model is clear from context, we drop the superscript $\mathcal{M}$, simply writing $\mathrm{SE}(\sigma)$ and $\mathrm{SE}(\pi)$.

Now, we have all the ingredients to define the semantics for $\mathsf{L}_{\mathsf{Kh}_i}$.

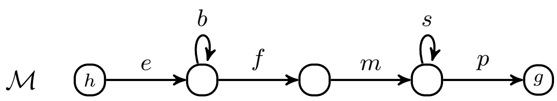
Definition 6. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$ and $w \in \mathrm{W}$. The satisfiability relation $\models$ for $\mathsf{L}_{\mathsf{Kh}_i}$ is inductively defined as:

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & p \in \mathrm{V}(w) \\
\mathcal{M}, w \models \neg\varphi & \text{iff} & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \psi \vee \varphi & \text{iff} & \mathcal{M}, w \models \psi \text{ or } \mathcal{M}, w \models \varphi \\
\mathcal{M}, w \models \mathsf{Kh}_i(\psi, \varphi) & \text{iff} & \text{there is } \pi \in \mathrm{U}(i) \text{ such that:} \\
 & & \quad \textbf{\textit{(i)}}\ [\![\psi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi), \text{ and} \\
 & & \quad \textbf{\textit{(ii)}}\ \mathrm{R}_\pi([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}},
\end{array}
$$

with $[\![\chi]\!]^{\mathcal{M}} := \{w \in \mathrm{W} \mid \mathcal{M}, w \models \chi\}$. Any π making true the existential statement in the semantic clause of $\mathsf{Kh}_i(\psi, \varphi)$ is called a *witness* for $\mathsf{Kh}_i(\psi, \varphi)$. Define $\mathcal{M} \models \varphi$ iff $[\![\varphi]\!]^{\mathcal{M}} = \mathrm{W}$, and $\models \varphi$ iff $\mathcal{M} \models \varphi$, for all $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}$. These notions are extended as expected for all the logics in the rest of the article.

Some comments are useful here. First note how the modality A (respectively, E), defined by abbreviation in Def. 1, is actually the universal (respectively, existential) *global modality* from, e.g., [37]. Indeed, for every model $\mathcal{M}$ and every state w, $\mathcal{M}, w \models \mathsf{A}\varphi$ ($\mathcal{M}, w \models \mathsf{E}\varphi$) holds if and only if φ is true in every (some) state in $\mathcal{M}$ (see [30, 31] for details). Second, notice that $\mathsf{L}_{\mathsf{Kh}_i}$ is a very 'simple' language. In particular, even though the Kh_i modality has fairly complex requirements for its witness π, the language is 'blind' to the actual actions that appear in the semantics. Indeed, as we are going to discuss below, the sets of plans in a given model can be drastically changed without affecting the agents' abilities as described by the language.

Example 7. Consider a simple scenario in which two agents i and j attempt to bake a good cake (represented by g). Suppose they follow a similar recipe, and they have all the ingredients (h). The recipe states that g is achieved via the following steps: adding eggs (e), beating the eggs (b), adding flour (f), adding milk (m), stirring these ingredients (s) and finally baking the preparation (p). Thus, the plan needed to achieve g is *ebfmsp*, whenever the agents have all the ingredients (h). Agent i, an experienced chef, is aware that this is the way to get a good cake. On the other hand, agent j has no cooking experience, so she considers that the order of the steps does not matter (e.g., she thinks she can add milk before adding the flour).



$\mathrm{U}(i) = \{\ \{ebfmsp\}\ \}$
$\mathrm{U}(j) = \{\ \{ebfmsp, ebmfsp\}\ \}$

The diagram shows, on the right, the set of indistinguishable plans in $\mathrm{U}(i)$ and in $\mathrm{U}(j)$. Agent i knows how to bake a good cake, provided she has all the ingredients (i.e., $\mathcal{M} \models \mathsf{Kh}_i(h, g)$). This is because agent i *distinguishes ebfmsp* as the "good plan". On the other hand, j considers that adding milk and adding flour can be done in any order (i.e., *ebfmsp* and *ebmfsp* are indistinguishable), leading to $\mathcal{M} \not\models \mathsf{Kh}_j(h, g)$, as the plan *ebmfsp* is not strongly executable in the model.

The notion of *bisimulation* is a crucial tool for understanding the expressive power of a modal language. Here we recall the notion of bisimulation for $\mathsf{L}_{\mathsf{Kh}_i}$ over $\mathrm{LTS}^{\mathrm{U}}$s [31]. We start by providing some useful abbreviations.

Definition 8. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$ over Prop, Act and Agt. Take $\pi \in 2^{(\mathsf{Act}^*)}$, $U, T \subseteq \mathrm{W}$ and $i \in \mathsf{Agt}$.

Axioms	Taut	$\vdash \varphi$ for φ a propositional tautology
	DistA	$\vdash \mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$
	TA	$\vdash \mathsf{A}\varphi \to \varphi$
	$\mathsf{4KhA}$	$\vdash \mathsf{Kh}_i(\psi,\varphi) \to \mathsf{A}\mathsf{Kh}_i(\psi,\varphi)$
	$\mathsf{5KhA}$	$\vdash \neg\mathsf{Kh}_i(\psi,\varphi) \to \mathsf{A}\neg\mathsf{Kh}_i(\psi,\varphi)$
	KhA	$\vdash (\mathsf{A}(\chi \to \psi) \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}(\varphi \to \theta)) \to \mathsf{Kh}_i(\chi,\theta)$
	G	$\vdash \mathsf{Kh}_i(\varphi,\bot) \to \mathsf{Kh}_j(\varphi,\bot)$
Rules	MP	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
	NecA	From $\vdash \varphi$ infer $\vdash \mathsf{A}\varphi$

Table 1: Axiomatization $\mathcal{L}_{\mathsf{Kh}_i}$ for $\mathsf{L}_{\mathsf{Kh}_i}$ w.r.t. $\mathrm{LTS}^{\mathrm{U}}$*s*.

- Write $U \overset{\pi}{\Rrightarrow} T$ *iff*$_{def}$ $U \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi(U) \subseteq T$.
- Write $U \overset{i}{\Rrightarrow} T$ *iff*$_{def}$ there is $\pi \in \mathrm{U}(i)$ such that $U \overset{\pi}{\Rrightarrow} T$.

Additionally, $U \subseteq \mathrm{W}$ is propositionally definable in $\mathcal{M}$ if and only if there is a propositional formula φ such that $U = [\![\varphi]\!]^{\mathcal{M}}$.

Now we are ready to introduce the notion of bisimulation for $\mathsf{L}_{\mathsf{Kh}_i}$.

Definition 9 ($\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation)**.** Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V}\rangle$ and $\mathcal{M}' = \langle \mathrm{W}', \mathrm{R}', \mathrm{U}', \mathrm{V}'\rangle$ be $\mathrm{LTS}^{\mathrm{U}}$*s*. A non-empty $Z \subseteq \mathrm{W} \times \mathrm{W}'$ is called an $\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ if and only if wZw' implies all of the following:

- **Atom**: $\mathrm{V}(w) = \mathrm{V}'(w')$.
- Kh_i**-Zig**: for any *propositionally* definable $U \subseteq \mathrm{W}$, if $U \overset{i}{\Rrightarrow} T$ for some $T \subseteq \mathrm{W}$, then there is $T' \subseteq \mathrm{W}'$ such that: 1) $Z(U) \overset{i}{\Rrightarrow} T'$, and 2) $T' \subseteq Z(T)$.
- Kh_i**-Zag**: for any *propositionally* definable $U' \subseteq \mathrm{W}'$, if $U' \overset{i}{\Rrightarrow} T'$ for some $T' \subseteq \mathrm{W}'$, then there is $T \subseteq \mathrm{W}$ such that: 1) $Z^{-1}(U') \overset{i}{\Rrightarrow} T$, and 2) $T \subseteq Z^{-1}(T')$.
- A**-Zig**: for all $u \in \mathrm{W}$ there is a $u' \in \mathrm{W}'$ such that uZu'.
- A**-Zag**: for all $u' \in \mathrm{W}'$ there is a $u \in \mathrm{W}$ such that uZu'.

We write $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ when there is an $\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that wZw'.

The following theorem establishes a classical adequacy result.

Theorem 10 ([31])**.** *Let $\mathcal{M}, w$ and $\mathcal{M}', w'$ be two $\mathrm{LTS}^{\mathrm{U}}$s. $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ implies $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$, for all $\mathsf{L}_{\mathsf{Kh}_i}$-formulas φ.*

Another important element to consider is the axiom system for $\mathsf{L}_{\mathsf{Kh}_i}$ [30, 31].

Theorem 11 ([30, 31])**.** *The axiom system from Table 1 is sound and strongly complete with respect to the class of all $\mathrm{LTS}^{\mathrm{U}}$s.*

It is worth noticing two differences between the system in Table 1 and the one presented in [31]. First, the formula $(\mathsf{E}\psi \wedge \mathsf{Kh}_i(\psi,\varphi)) \to \mathsf{E}\varphi$, called KhE and part of the axiom system presented in [31], is omitted here. This is because it is derivable in the system, as proved below by showing the derivability of the equivalent formula $(\mathsf{E}\psi \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}\neg\varphi) \to \bot$. One can start with

$$\begin{array}{lll} & \vdash (\mathsf{E}\psi \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}\neg\varphi) \to (\mathsf{E}\psi \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}\neg\varphi) & \mathsf{TAUT} \\ \Leftrightarrow & \vdash (\mathsf{E}\psi \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}\neg\varphi) \to (\mathsf{E}\psi \wedge \mathsf{A}(\psi \to \psi) \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}(\varphi \to \bot)) & \mathsf{TAUT}, \mathsf{NECA} \end{array}$$

Then, by using an instance of KhA, we have that

$$\begin{array}{lll} & \vdash (\mathsf{A}(\psi \to \psi) \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}(\varphi \to \bot)) \to (\mathsf{Kh}_i(\psi,\bot)) & \\ \Leftrightarrow & \vdash (\mathsf{E}\psi \wedge \mathsf{A}(\psi \to \psi) \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}(\varphi \to \bot)) \to (\mathsf{E}\psi \wedge \mathsf{Kh}_i(\psi,\bot)) & \mathsf{TAUT} \\ \Leftrightarrow & \vdash (\mathsf{E}\psi \wedge \mathsf{A}(\psi \to \psi) \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}(\varphi \to \bot)) \to (\neg \mathsf{A}\neg\psi \wedge \mathsf{A}\neg\psi) & \text{Def. } \mathsf{A}, \mathsf{E} \\ \Leftrightarrow & \vdash (\mathsf{E}\psi \wedge \mathsf{A}(\psi \to \psi) \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}(\varphi \to \bot)) \to \bot & \text{Def. } \bot \end{array}$$

Given the implications proved above, KhE follows by transitivity.

The second difference is the presence of the axiom G (standing for 'global' formulas). This axiom indicates that the validity of formulas of the form $\mathsf{Kh}_i(\varphi,\bot)$ is shared among all agents. Notice that these formulas are particular, as they describe conditions that would lead to the impossible case of a goal satisfying $\bot$. The axiom states that these formulas do not depend on the particular agent under consideration. This axiom needs to be added as a consequence of a gap in the arguments in [31, Prop. 4]. The validity called SCond cannot be proved without axiom G (since the universal modality A is defined in that article as a disjunction, the last step in [31, Prop. 4] is not an equivalence). By adding G here, we fix this issue.

Strong completeness of the axiomatization above is established in [30, 31] via a canonical model construction (modulo the correction just mentioned). In addition, starting from the canonical model, a careful selection function can be used to prove a small (polynomial) model property that leads to the following complexity results.

Theorem 12 ([30, 31])**.** *The model-checking problem for $\mathsf{L}_{\mathsf{Kh}_i}$ is in P, while the satisfiability problem for $\mathsf{L}_{\mathsf{Kh}_i}$ is NP-complete.*

We mentioned before that the semantic interpretation for Kh_i is 'blind' to certain aspects of its witness. In fact, as the completeness proof in [30] shows, it cannot distinguish between the class of all arbitrary $\mathrm{LTS}^{\mathrm{U}}$ and the class in which, for every agent i, every set in $\mathrm{U}(i)$ is a set of one-step plans (i.e., a set of basic actions). Formally, we can define this class as follows:

Definition 13. Define $\mathbf{M_{BA}}$ (**BA** stands for 'basic actions') as the class of models $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ in which, for all $i \in \mathsf{Agt}$, we have that $\pi \in \mathrm{U}(i)$ implies $\pi \subseteq \mathsf{Act}$.

$\mathbf{M_{BA}}$ (denoted $\mathbf{M}^1$ in [34]) is a restricted class of models, which could be interpreted as a more abstract representation of the actions available to agents. In this class, every plan is modeled as a single atomic action (similar to what is done in formal verification using the so-called 'path abstraction' technique; see, e.g., [38] for an example).

Proposition 14. *A formula in $\mathsf{L}_{\mathsf{Kh}_i}$ is valid over the class of all* $\mathrm{LTS}^{\mathrm{U}}$ *if and only if it is valid over $\mathbf{M_{BA}}$.*

The last proposition can be interpreted as a limitation on the expressivity of $\mathsf{L}_{\mathsf{Kh}_i}$: since actions do not appear in its syntax, the logic is unable to distinguish between models in which witnesses are arbitrary plans and those in which they are all single atomic actions.

After recalling the previous basic definitions and useful results of the uncertainty-based semantics for knowing how, it is time to discuss dynamic operators. The underlying idea is that of DEL: updates in the model are linked to model-changing operations, and their effect can be described by modalities whose semantic interpretation relies not only on the initial model, but also on the modified one.

In an LTS^{U} $\langle \text{W}, \text{R}, \text{U}, \text{V} \rangle$ there is a clear distinction between ontic and epistemic information. On the one hand, the underlying LTS $\langle \text{W}, \text{R}, \text{V} \rangle$ provides *ontic/objective* facts indicating what the actions themselves (and the plans derived from them) can do. On the other hand, the indistinguishability sets in U describe the *epistemic* state of the agents, indicating which plans are available and the level to which the agents can discern among them. Thus, while changes in the underlying LTS can be seen as changes in a 'dynamic' world to which the agents react by adjusting their epistemic state accordingly (analogous to *belief update* in the belief change literature [39] and to ontic changes in DEL [40]), changes in the indistinguishability sets can be seen as changes in the agents' capacity to discern the world (analogous to *belief revision* in the belief change literature and to epistemic changes in DEL).

The following sections explore different model-changing operations one can perform over LTS^{U}s, discussing their interpretations as well as technical results. While some examples of ontic changes will be discussed in Sec. 3, the main focus of the article is on epistemic updates (Secs. 4 and 5).

3 Ontic updates

This section discusses model operations updating the underlying LTS in an LTS^{U}. We discuss operations that remove states (possibly changing also the accessibility relations) and operations that delete edges directly. Within the EL setting, these two operations have a natural interpretation: they represent ways in which an agent's uncertainty changes (in this sense, the second can be seen as a generalization of the first). Here, they are reinterpreted to match what the affected model components (domain and relations) represent in the indistinguishability-based knowing how setting. In each case, we present a discussion, a modality for describing the effects of the operation, and a sound and complete reduction-axioms-based axiomatization over a particular class of models.

3.1 Removing states

A well-established form of updating relational models is the elimination of worlds. This model update is interpreted as a truthful announcement indicating to all agents, in a fully public way, that a formula χ is the case [12, 35]. To describe the effects of this action, Public Announcement Logic (PAL) extends the basic modal logic with a dynamic operator $[\chi]$. Given an LTS $\mathcal{M} = \langle \text{W}, \text{R}, \text{V} \rangle$, the semantic interpretation of $[\chi]$ is usually defined as

$$\mathcal{M}, w \models [\chi]\varphi \quad \text{iff} \quad \mathcal{M}, w \models \chi \text{ implies } \mathcal{M}_\chi, w \models \varphi,$$

with the LTS $\mathcal{M}_\chi = \langle \text{W}_\chi, \text{R}_\chi, \text{V}_\chi \rangle$ given by $\text{W}_\chi = \{w \mid \mathcal{M}, w \models \chi\}$, $(\text{R}_\chi)_a = \text{R}_a \cap \text{W}_\chi^2$ for each $a \in \mathsf{Act}$, and $\text{V}_\chi(w) = \text{V}(w)$ for all $w \in \text{W}_\chi$. In other words, $\mathcal{M}_\chi$ is obtained by taking $[\![\chi]\!]^{\mathcal{M}}$ as the new domain, then restricting relations and valuation accordingly. This operation can be naturally extended to $\mathrm{LTS}^{\mathrm{U}}$s by setting $\mathrm{U}_{\chi} = \mathrm{U}$ to indicate that the update does not affect indistinguishability sets.

A remark might be useful here. In the original knowing how setting with semantics defined in terms of LTSs [24], edges can be taken to encode the agents' *epistemic take on their abilities*: the actions they consider possible, the preconditions required for their execution, and the effects they have (how a situation changes under the execution of a given action). As a result, in such a setting, updates on states and edges are epistemic updates. In the $\mathrm{LTS}^{\mathrm{U}}$-based semantics, however, the edges in an $\mathrm{LTS}^{\mathrm{U}}$ stand for the actions available in the considered situation, their preconditions and effects, independently of the agents' knowledge about them. In short, the underlying LTS in an $\mathrm{LTS}^{\mathrm{U}}$ provides ontic information. Thus, removing states and edges produces an *ontic* change that might affect, indirectly, the agents' epistemic abilities. The elimination of some states indicates not that they are no longer (epistemically) *possible*, but rather that they are no longer existent or (action-wise) *reachable* in the current situation.

It can be shown that the update operator adds expressivity to $\mathsf{L}_{\mathsf{Kh}_i}$ (a similar result was established in [24] for the original LTS-based setting).

Proposition 15. *Adding $[\chi]$ to $\mathsf{L}_{\mathsf{Kh}_i}$ increases the expressive power.*

Proof. Consider the pointed $\mathrm{LTS}^{\mathrm{U}}$s $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ below, with $\mathrm{U}(i) = \mathrm{U}'(i) = \{\{a\}\}$, and with dashed lines indicating the nodes and edges removed by an update with $[p]$.

These pointed models are $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar (Def. 9), and thus indistinguishable in $\mathsf{L}_{\mathsf{Kh}_i}$. However, they can be distinguished by a formula using the update modality: $\mathcal{M}, w \models [p]\mathsf{Kh}_i(p, q)$ and yet $\mathcal{M}', w' \not\models [p]\mathsf{Kh}_i(p, q)$. ☐

A consequence of Prop. 15 is that the modality for PAL-like updates $[\chi]$ is not reducible to the underlying language $\mathsf{L}_{\mathsf{Kh}_i}$. This makes sense: while $\mathsf{L}_{\mathsf{Kh}_i}$ can express the agents' abilities to achieve certain goals from given situations (the modality Kh_i), it cannot talk about more specific properties of the courses of action on which the abilities rely. This is in contrast with what happens when these modalities are added to standard epistemic logic, where reduction axioms can be defined (see, e.g., [12]). In such situations, the axiomatization can be approached in different ways. For this operation, the 'extend the basic language' method has been discussed in [24], but no successful extension was found. Here, the strategy is to look for a variation of the update operation, and then to restrict the setting to a particular class of models.

Definition 16. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$; let χ be a formula. The $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}_{!\chi} = \langle \mathrm{W}_{!\chi}, \mathrm{R}_{!\chi}, \mathrm{U}_{!\chi}, \mathrm{V}_{!\chi} \rangle$ is given by:
- $\mathrm{W}_{!\chi} = [\![\chi]\!]^{\mathcal{M}}$,
- $(\mathrm{R}_{!\chi})_a = \{(w, v) \in \mathrm{R}_a \mid w \in [\![\chi]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \subseteq [\![\chi]\!]^{\mathcal{M}}\}$ for every $a \in \mathsf{Act}$,
- $\mathrm{U}_{!\chi} = \mathrm{U}$, and

- $V_{!\chi}(w) = V(w)$ for all $w \in W_{!\chi}$.

The language $\mathsf{PAL}_{\mathsf{Kh}_i}$ extends $\mathsf{L}_{\mathsf{Kh}_i}$ with formulas of the form $[!\chi]\varphi$, with the semantic interpretation of the new formulas given by

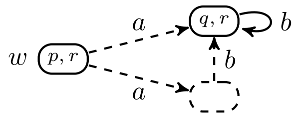
$$\mathcal{M}, w \models [!\chi]\varphi \quad \textit{iff} \quad \mathcal{M}, w \models \chi \text{ implies } \mathcal{M}_{!\chi}, w \models \varphi.$$

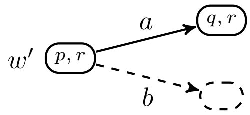
Thus, given a model $\mathcal{M}$, the only difference between the standard $\mathcal{M}_\chi$ and the just defined $\mathcal{M}_{!\chi}$ is in the definition of its relations. Indeed, in the former, each $(R_\chi)_a$ restricts the original R_a to the new domain. In the latter, however, each $(R_{!\chi})_a$ is defined point-wise: it is exactly as $R_a(w)$ if w and all the states $R_a(w)$ can reach will survive the operation ($R_a(w) \cup \{w\} \subseteq [\![\chi]\!]^{\mathcal{M}}$ implies $(R_{!\chi})_a(w) = R_a(w)$), and is empty otherwise ($R_a(w) \cup \{w\} \not\subseteq [\![\chi]\!]^{\mathcal{M}}$ implies $(R_{!\chi})_a(w) = \emptyset$).

Even with this more restricted version of an update, the resulting logic fails to have reduction axioms: adding $[!\chi]$ to $\mathsf{L}_{\mathsf{Kh}_i}$ increases the expressive power.

Proposition 17. *$\mathsf{PAL}_{\mathsf{Kh}_i}$ is more expressive than $\mathsf{L}_{\mathsf{Kh}_i}$ over arbitrary $\mathrm{LTS}^{\mathrm{U}}$s.*

Proof. Let $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ be the single agent pointed models depicted below, with $U(i) := \{\{ab\}\}$ and $U'(i) := \{\{a\}\}$:

$\mathcal{M}$ 

 $\mathcal{M}'$

The pointed models are $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar (Def. 9), hence, they satisfy the same formulas in $\mathsf{L}_{\mathsf{Kh}_i}$. However, $\mathcal{M}, w \not\models [!r]\mathsf{Kh}_i(p, q)$ (given that $\mathcal{M}, w \models r$ and $\mathcal{M}_{!r}, w \not\models \mathsf{Kh}_i(p, q)$) whereas $\mathcal{M}', w' \models [!r]\mathsf{Kh}_i(p, q)$ (given that $\mathcal{M}', w' \models r$ and $\mathcal{M}'_{!r}, w \models \mathsf{Kh}_i(p, q)$). $\square$

In the proposition above, both models $\mathcal{M}$ and $\mathcal{M}'$ satisfy the formula $\mathsf{Kh}_i(p, q)$, with their respective witnesses being $\{ab\}$ and $\{a\}$. Still, as discussed, after eliminating worlds in which r is false, this is no longer the case: the formula fails in $\mathcal{M}_{!r}$ but holds in $\mathcal{M}'_{!r}$. Notice that the witness in $\mathcal{M}$ is a singleton set with a two-step plan, while the witness in $\mathcal{M}'$ is a singleton set with a one-step plan (i.e., an atomic action). This makes a difference because, by removing intermediate steps (which exist in the first but not in the second), $\mathsf{PAL}_{\mathsf{Kh}_i}$ can tell these two models apart.

Now, recall the class $\mathbf{M_{BA}}$ introduced in Def. 13. By restricting the class of models to $\mathbf{M_{BA}}$, one can still get agents with 'the same abilities' while also stopping $\mathsf{PAL}_{\mathsf{Kh}_i}$ from being able to tell $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar models apart. Indeed, the reduction axioms from Table 2 are valid in the class of models $\mathbf{M_{BA}}$. Since we can use these axioms to eliminate announcements by iteratively replacing the innermost occurrence of a $[!\chi]$ modality, completeness for $\mathsf{PAL}_{\mathsf{Kh}_i}$ over $\mathbf{M_{BA}}$ follows.

To establish the soundness of the reduction axioms, we first need to prove the following semantic properties.

Lemma 18. *Let χ and φ be $\mathsf{PAL}_{\mathsf{Kh}_i}$-formulas and let $\mathcal{M}$ be an arbitrary $\mathrm{LTS}^{\mathrm{U}}$. The following equalities hold:*

RAtom	$\vdash [!\chi]p \leftrightarrow (\chi \to p), p \in \mathsf{Prop}$
R$\neg$	$\vdash [!\chi]\neg\varphi \leftrightarrow (\chi \to \neg[!\chi]\varphi)$
R$\vee$	$\vdash [!\chi](\varphi \vee \psi) \leftrightarrow [!\chi]\varphi \vee [!\chi]\psi$
RKh	$\vdash [!\chi]\mathsf{Kh}_i(\varphi,\psi) \leftrightarrow (\chi \to \mathsf{Kh}_i(\chi \wedge [!\chi]\varphi, \chi \wedge [!\chi]\psi))$
RE$_{[!]}$	From $\vdash \varphi \leftrightarrow \psi$ derive $\vdash [!\chi]\varphi \leftrightarrow [!\chi]\psi$

Table 2: Reduction axioms $\mathcal{L}_{\mathsf{PAL}_{\mathsf{Kh}_i}}$.

1. $[\![[!\chi]\varphi]\!]^{\mathcal{M}} = [\![\neg\chi]\!]^{\mathcal{M}} \cup [\![\varphi]\!]^{\mathcal{M}_{!\chi}}$. *2.* $[\![\varphi]\!]^{\mathcal{M}_{!\chi}} = [\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}$.

Proof. For Item 1, note that $w \in [\![[!\chi]\varphi]\!]^{\mathcal{M}}$ iff $\mathcal{M}, w \models [!\chi]\varphi$, and thus iff $\mathcal{M}, w \models \chi$ implies $\mathcal{M}_{!\chi}, w \models \varphi$. The latter is equivalent to $\mathcal{M}, w \models \neg\chi$ or $\mathcal{M}_{!\chi}, w \models \varphi$, so $w \in [\![\neg\chi]\!]^{\mathcal{M}} \cup [\![\varphi]\!]^{\mathcal{M}_{!\chi}}$.

For Item 2, $w \in [\![\varphi]\!]^{\mathcal{M}_{!\chi}}$ iff $w \in \mathrm{W}_{!\chi} = [\![\chi]\!]^{\mathcal{M}}$ and $w \in [\![\varphi]\!]^{\mathcal{M}_{!\chi}}$, and thus iff $w \in [\![\chi]\!]^{\mathcal{M}} \cap [\![\varphi]\!]^{\mathcal{M}_{!\chi}}$. The previous set is in the form of $(A \cap B)$ so, by set reasoning, it is equal to $(A \cap (A^c \cup B))$. Thus, the statement above is equivalent to $w \in [\![\chi]\!]^{\mathcal{M}} \cap ([\![\neg\chi]\!]^{\mathcal{M}} \cup [\![\varphi]\!]^{\mathcal{M}_{!\chi}})$ which holds (the result of the first item) iff $w \in [\![\chi]\!]^{\mathcal{M}} \cap [\![[!\chi]\varphi]\!]^{\mathcal{M}}$, i.e., iff $w \in [\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}$. □

Lemma 19. *The reduction axioms from Table 2 are sound in* $\mathbf{M_{BA}}$.

Proof. We will focus only on axiom RKh, while proving the validity of the rest can be done as in PAL.

By the definition of $[!\chi]$, $\mathcal{M}, w \models [!\chi]\mathsf{Kh}_i(\varphi,\psi)$ iff $\mathcal{M}, w \models \chi$ implies $\mathcal{M}_{!\chi}, w \models \mathsf{Kh}_i(\varphi,\psi)$. Using the definition of Kh_i, $\mathcal{M}_{!\chi}, w \models \mathsf{Kh}_i(\varphi,\psi)$ iff there is $\pi \in (\mathrm{U}_{!\chi})(i)$ with $\pi \subseteq \mathsf{Act}$ s.t. $[\![\varphi]\!]^{\mathcal{M}_{!\chi}} \subseteq \mathrm{SE}^{\mathcal{M}_{!\chi}}(\pi)$ and $(\mathrm{R}_{!\chi})_\pi([\![\varphi]\!]^{\mathcal{M}_{!\chi}}) \subseteq [\![\psi]\!]^{\mathcal{M}_{!\chi}}$. By the definition of $\mathcal{M}_{!\chi}$, $(\mathrm{U}_{!\chi})(i) = \mathrm{U}(i)$. Using Lemma 18, $[\![\varphi]\!]^{\mathcal{M}_{!\chi}} = [\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}$ and $[\![\psi]\!]^{\mathcal{M}_{!\chi}} = [\![\chi \wedge [!\chi]\psi]\!]^{\mathcal{M}}$. Thus, $\mathcal{M}_{!\chi}, w \models \mathsf{Kh}_i(\varphi,\psi)$ iff there is $\pi \in \mathrm{U}(i)$ with $\pi \subseteq \mathsf{Act}$ s.t. $[\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}^{\mathcal{M}_{!\chi}}(\pi)$ and $(\mathrm{R}_{!\chi})_\pi([\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}) \subseteq [\![\chi \wedge [!\chi]\psi]\!]^{\mathcal{M}}$. Let $a \in \pi$ and $w \in \mathrm{W}_{!\chi}$. If $w \in [\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}$, then:

- $(\mathrm{R}_{!\chi})_a(w) \neq \emptyset$ (since $w \in \mathrm{SE}^{\mathcal{M}_{!\chi}}(a)$) and
- $(\mathrm{R}_{!\chi})_a(w) \subseteq [\![\chi \wedge [!\chi]\psi]\!]^{\mathcal{M}}$.

Using Def. 16, the first item is equivalent to $w \in [\![\chi]\!]^{\mathcal{M}}$, $\mathrm{R}_a(w) \subseteq [\![\chi]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \neq \emptyset$ and with this information, $(\mathrm{R}_{!\chi})_a(w) = \mathrm{R}_a(w)$ that is useful for the second item. Hence, if $w \in [\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}$, then:

- $w \in [\![\chi]\!]^{\mathcal{M}}$, $\mathrm{R}_a(w) \subseteq [\![\chi]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \neq \emptyset$ and
- $\mathrm{R}_a(w) \subseteq [\![\chi \wedge [!\chi]\psi]\!]^{\mathcal{M}}$.

Note that $w \in [\![\chi]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \subseteq [\![\chi]\!]^{\mathcal{M}}$ are redundant as $w \in [\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \subseteq [\![\chi \wedge [!\chi]\psi]\!]^{\mathcal{M}}$. With this, if $w \in [\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}$, then:

- $\mathrm{R}_a(w) \neq \emptyset$ (thus, $w \in \mathrm{SE}^{\mathcal{M}}(a)$) and
- $\mathrm{R}_a(w) \subseteq [\![\chi \wedge [!\chi]\psi]\!]^{\mathcal{M}}$.

Since we prove for arbitrary $a \in \pi$ and $w \in \mathrm{W}_{!\chi}$, the result yields for all $a \in \pi$ and $w \in \mathrm{W}_{!\chi}$. Moreover, it can be generalized for all $w \in \mathrm{W}$ as if $w \in [\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}$, then $w \in \mathrm{W}_{!\chi}$. Now $\mathcal{M}_{!\chi}, w \models \mathsf{Kh}_i(\varphi,\psi)$ iff there is $\pi \in \mathrm{U}(i)$ with $\pi \subseteq \mathsf{Act}$ s.t. $[\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi([\![\chi \wedge [!\chi]\varphi]\!]^{\mathcal{M}}) \subseteq [\![\chi \wedge [!\chi]\psi]\!]^{\mathcal{M}}$. This happens iff $\mathcal{M}, w \models \mathsf{Kh}_i(\chi \wedge [!\chi]\varphi, \chi \wedge [!\chi]\psi)$. Since for this equivalence we prove assuming $\mathcal{M}, w \models \chi$, then $\mathcal{M}, w \models \chi$ implies $\mathcal{M}_{!\chi}, w \models \mathsf{Kh}_i(\varphi, \psi)$ iff $\mathcal{M}, w \models \chi$ implies $\mathcal{M}, w \models \mathsf{Kh}_i(\chi \wedge [!\chi]\varphi, \chi \wedge [!\chi]\psi)$. Thus, iff $\mathcal{M}, w \models \chi \to \mathsf{Kh}_i(\chi \wedge [!\chi]\varphi, \chi \wedge [!\chi]\psi)$. ☐

Since all the reduction axioms are sound, we can state the intended result.

Theorem 20. *$\mathcal{L}_{\mathsf{Kh}_i}$ plus the reduction axioms for $[!\chi]$ in Table 2 form a sound and strongly complete axiomatization for $\mathsf{PAL}_{\mathsf{Kh}_i}$ with respect to $\mathbf{M_{BA}}$.*

Proof. The result follows by the correctness of the axioms and rule from Table 2 stated in Lemma 19 (which enables us to eliminate all the occurences of a $[!\chi]$ modality), together with the fact that the system in Table 1 is also complete with respect to $\mathbf{M_{BA}}$ (see the proof in [30, 31]). ☐

It is worth noticing that, even in the class $\mathbf{M_{BA}}$, the first discussed modality, $[\chi]$, adds expressive power to $\mathsf{L}_{\mathsf{Kh}_i}$. Thus, there are no reduction axioms to eliminate its occurrences in formulas. This follows from Prop. 15, since the counterexample models in the proof belong to the class $\mathbf{M_{BA}}$. Therefore, the meaning of $[\chi]$ and $[!\chi]$ is different even in such a restricted class of models.

To finish this discussion, we prove that the satisfiability problem for $\mathsf{PAL}_{\mathsf{Kh}_i}$ is decidable over $\mathbf{M_{BA}}$.

Corollary 21. *The satisfiability problem for $\mathsf{PAL}_{\mathsf{Kh}_i}$ over $\mathbf{M_{BA}}$ is decidable.*

Proof. Let φ be a formula of $\mathsf{PAL}_{\mathsf{Kh}_i}$. By using the reduction axioms from Table 2 repeatedly, φ can be translated (in a finite number of steps) into a formula φ', such that φ and φ' are equivalent in the class $\mathbf{M_{BA}}$ (Lemma 19). Since the satisfiability problem for $\mathsf{L}_{\mathsf{Kh}_i}$ is decidable ([30, 31]), we can compute the satisfiability of φ'. There are two possible outcomes: ***(i)*** If φ' is satisfiable, then it is satisfiable in $\mathbf{M_{BA}}$. This is the case because [30, 31] shows that every formula in $\mathsf{L}_{\mathsf{Kh}_i}$ is satisfiable if it is satisfiable in a finite model where each $\mathrm{U}(i) \subseteq \{\{a\} \mid a \in \mathsf{Act}\}$, a subclass of models strictly contained in $\mathbf{M_{BA}}$. ***(ii)*** If φ' is not satisfiable over LTSs then, a fortiori, it is not satisfiable in $\mathbf{M_{BA}}$. ☐

3.2 Removing edges

Another way to update LTSs is by modifying their relations directly, while keeping the domain intact. In the context of DEL, one proposal that stands out is Arrow Update Logic (AUL) [36]. The dynamic operator of AUL takes as a parameter a list of arrow specifications in the form of triples (formula, action, formula). Then, an edge with label a from state w to state v will not be deleted if and only if there is an arrow specification $(\varphi_1, a, \varphi_2)$ such that w satisfies φ_1 and v satisfies φ_2.

The version of the AUL framework presented below differs from the original in two aspects. First, an arrow specification is defined as a pair (formula, formula) with no reference to actions, in line with the syntax of L_{Kh} which does not include them. Second, as in Def. 16, an edge labeled a from state w to state v will not be deleted if there exists an arrow specification (φ_1, φ_2) such that w satisfies φ_1 and *all states* reachable from w via a-edges satisfy φ_2. The reason for this modification is similar to the one we discussed for the $[\chi]$ operator. The original AUL modality adopts an

RJoin	$[S]\varphi \leftrightarrow [(\bigwedge_{i=1}^{n} \theta_i, \bigwedge_{i=1}^{n} \theta'_i)]\varphi$
RAtom	$[(\theta, \theta')]p \leftrightarrow p, p \in \mathsf{Prop}$
R$\neg$	$[(\theta, \theta')]\neg\varphi \leftrightarrow \neg[(\theta, \theta')]\varphi$
R$\vee$	$[(\theta, \theta')](\varphi \vee \psi) \leftrightarrow [(\theta, \theta')]\varphi \vee [(\theta, \theta')]\psi$
RKh	$[(\theta, \theta')]\mathsf{Kh}_i(\varphi, \psi) \leftrightarrow \mathsf{A}([(\theta, \theta')]\varphi \to \theta) \wedge \mathsf{Kh}_i([(\theta, \theta')]\varphi, \theta' \wedge [(\theta, \theta')]\psi)$
RE$_U$	From $\vdash \varphi \leftrightarrow \psi$ derive $\vdash [(\theta, \theta')]\varphi \leftrightarrow [(\theta, \theta')]\psi$

Table 3: Reduction axioms $\mathcal{L}_{\mathsf{AUL}_{\mathsf{Kh}_i}}$, with $S = (\theta_1, \theta'_1), \ldots, (\theta_n, \theta'_n)$.

edge-by-edge perspective rather than a group-edge one, as is done in the semantics of Kh_i-formulas.

Definition 22. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V}\rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$; let $S = (\theta_1, \theta'_1), \ldots, (\theta_n, \theta'_n)$ be a finite list of arrow specifications with θ_i, θ'_i formulas for every $0 \leq i \leq n$. The $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}_S = \langle \mathrm{W}, \mathrm{R}_S, \mathrm{U}, \mathrm{V}\rangle$ is such that, for every $a \in \mathsf{Act}$, $(\mathrm{R}_S)_a = \{(w, v) \in \mathrm{R}_a(w) \mid w \in [\![\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}},\ \mathrm{R}_a(w) \subseteq [\![\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}\}$.

In the definition above, notice that if $w \in [\![\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \subseteq [\![\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}$, then $(\mathrm{R}_S)_a(w) = \mathrm{R}_a(w)$. Moreover, $(\mathrm{R}_S)_a(w) \neq \emptyset$ iff $w \in [\![\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}}$, $\mathrm{R}_a(w) \subseteq [\![\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \neq \emptyset$. We are now ready to introduce the language.

Definition 23. Formulas of the language $\mathsf{AUL}_{\mathsf{Kh}_i}$ are defined by the following grammar:

$$\begin{aligned} \varphi &::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi) \mid [S]\varphi, \\ S &::= (\varphi, \varphi) \mid S, (\varphi, \varphi), \end{aligned}$$

with $p \in \mathsf{Prop}$ and $i \in \mathsf{Agt}$. For the semantics, formulas already in $\mathsf{L}_{\mathsf{Kh}_i}$ are interpreted as before. For the new type of formulas,

$$\mathcal{M}, w \models [S]\varphi \text{ iff } \mathcal{M}_S, w \models \varphi.$$

Again, adding the defined arrow-update modalities increases the expressivity of the language.

Proposition 24. *$\mathsf{AUL}_{\mathsf{Kh}_i}$ is more expressive than $\mathsf{L}_{\mathsf{Kh}_i}$ over arbitrary $\mathrm{LTS}^{\mathrm{U}}$s.*

Proof. By using the models from Prop. 17, we have that $\mathcal{M}, w \not\models [(r, r)]\mathsf{Kh}_i(p, q)$ and $\mathcal{M}', w' \models [(r, r)]\mathsf{Kh}_i(p, q)$. □

As it was the case with $\mathsf{PAL}_{\mathsf{Kh}_i}$, if we restrict ourselves to models in $\mathbf{M_{BA}}$, there are reduction axioms (see Table 3) that can be used to eliminate all occurrences of the $[S]$ modality. As our presentation is a variation of the original AUL, the axiom system is novel. However, it follows standard ideas, and the soundness of most axioms is clear. Something worth noting is that axiom RJoin reflects exactly the semantics of $[S]$: if we have a finite list of pairs of formulas in an update specification S, we can group all formulas into a pair (θ, θ').

Lemma 25. *The reduction axioms from Table 3 are sound in $\mathbf{M_{BA}}$.*

Proof. We only discuss RKh. Let $S = (\theta, \theta')$, by the definition of $[S]$, $\mathcal{M}, w \models [S]\mathsf{Kh}_i(\varphi, \psi)$ iff $\mathcal{M}_S, w \models \mathsf{Kh}_i(\varphi, \psi)$. Using the definition of Kh_i, $\mathcal{M}_S, w \models \mathsf{Kh}_i(\varphi, \psi)$ iff there is $\pi \in \mathrm{U}(i)$ with $\pi \subseteq \mathsf{Act}$ s.t. $[\![\varphi]\!]^{\mathcal{M}_S} \subseteq \mathrm{SE}^{\mathcal{M}_S}(\pi)$ and $(\mathrm{R}_S)_\pi(w)([\![\varphi]\!]^{\mathcal{M}_S}) \subseteq$

$[\![\psi]\!]^{\mathcal{M}_S}$. Using the definition of $[S]$, $[\![\varphi]\!]^{\mathcal{M}_S} = [\![[S]\varphi]\!]^{\mathcal{M}}$ and $[\![\psi]\!]^{\mathcal{M}_S} = [\![[S]\psi]\!]^{\mathcal{M}}$. Thus, $\mathcal{M}_S, w \models \mathsf{Kh}_i(\varphi, \psi)$ iff there is $\pi \in \mathrm{U}(i)$ with $\pi \subseteq \mathsf{Act}$ s.t. $[\![[S]\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}^{\mathcal{M}_S}(\pi)$ and $(\mathrm{R}_S)_\pi(w)([\![[S]\varphi]\!]^{\mathcal{M}}) \subseteq [\![[S]\psi]\!]^{\mathcal{M}}$. Let $a \in \pi$ and $w \in \mathrm{W}$. If $w \in [\![[S]\varphi]\!]^{\mathcal{M}}$, then:

- $(\mathrm{R}_S)_a(w) \neq \emptyset$ (since $w \in \mathrm{SE}^{\mathcal{M}_S}(a)$) and
- $(\mathrm{R}_S)_a(w) \subseteq [\![[S]\psi]\!]^{\mathcal{M}}$.

By Def. 23, the first item is equivalent to $w \in [\![\theta]\!]^{\mathcal{M}}$, $\mathrm{R}_a(w) \subseteq [\![\theta']\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \neq \emptyset$ and hence $(\mathrm{R}_S)_a(w) = \mathrm{R}_a(w)$, useful for the second item. If $w \in [\![[S]\varphi]\!]^{\mathcal{M}}$ then:

- $w \in [\![\theta]\!]^{\mathcal{M}}$,
- $\mathrm{R}_a(w) \neq \emptyset$ (thus, $w \in \mathrm{SE}^{\mathcal{M}}(a)$),
- $\mathrm{R}_a(w) \subseteq [\![\theta']\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \subseteq [\![[S]\psi]\!]^{\mathcal{M}}$.

Since the first item is independent of π, we can put it outside the expression. It is easy to see now that $\mathcal{M}, w \models \mathsf{A}([S]\varphi \to \theta)$ and if $w \in [\![[S]\varphi]\!]^{\mathcal{M}}$, then:

- $\mathrm{R}_a(w) \neq \emptyset$ (thus, $w \in \mathrm{SE}^{\mathcal{M}}(a)$),
- $\mathrm{R}_a(w) \subseteq [\![\theta' \wedge [S]\psi]\!]^{\mathcal{M}}$.

Since $a \in \pi$ and $w \in \mathrm{W}$ were arbitrary, the result yields for all $a \in \pi$ and $w \in \mathrm{W}$. Now $\mathcal{M}_S, w \models \mathsf{Kh}_i(\varphi, \psi)$ iff $\mathcal{M}, w \models \mathsf{A}([S]\varphi \to \theta)$ and there is $\pi \in \mathrm{U}(i)$ with $\pi \subseteq \mathsf{Act}$ s.t. $[\![[S]\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi([S]\varphi) \subseteq [\![\theta' \wedge [S]\psi]\!]^{\mathcal{M}}$. This happens iff $\mathcal{M}, w \models (\mathsf{A}([S]\varphi \to \theta) \wedge \mathsf{Kh}_i([S]\varphi, \theta' \wedge [S]\psi))$ with $S = (\theta, \theta')$. □

As a corollary, we obtain the intended result.

Theorem 26. *$\mathcal{L}_{\mathsf{Kh}_i}$ together with the reduction axioms for $[S]$ in Table 3 are a sound and strongly complete axiomatization for $\mathsf{AUL}_{\mathsf{Kh}_i}$ w.r.t. $\mathbf{M_{BA}}$.*

Proof. Similar to Thm. 20. □

Again, if we consider the AUL modality from the literature instead of the version introduced here, we increase the expressive power of the logic $\mathsf{L}_{\mathsf{Kh}_i}$, even over the class $\mathbf{M_{BA}}$. This easily follows from Prop. 15 as for the case of PAL, by taking the formula $[(p,p)]\mathsf{Kh}_i(p,q)$ (or $[(p,a,p)]\mathsf{Kh}_i(p,q)$ in the version that also specifies the edge name) to distinguish the models used there. Further details are omitted here.

We finish this section with a last result about the computational behavior of $\mathsf{AUL}_{\mathsf{Kh}_i}$ over $\mathbf{M_{BA}}$.

Corollary 27. *The satisfiability problem for $\mathsf{AUL}_{\mathsf{Kh}_i}$ over $\mathbf{M_{BA}}$ is decidable.*

Proof. Similar to Cor. 21 (using Lemma 25). □

4 Epistemic updates: a first attempt

This section presents a first attempt at defining *epistemic* updates over knowing how. No complete axiomatization is available yet for the operators to be introduced. Instead, we discuss their expressivity, some interesting properties, and the challenges in obtaining complete axiomatic systems.

4.1 Removing uncertainty between two plans

As it has been discussed, $\mathrm{LTS}^{\mathrm{U}}$s allow a natural representation of the actions that affect the abilities of an agent as well as those affecting her epistemic state. In an

$\mathrm{LTS^U}$, the crucial epistemic components are the sets $\mathrm{U}(i)$, defining the plans agent i is aware of, and the degree at which she can discern among them. Thus, we can represent changes in the epistemic state of an agent by means of operations that modify $\mathrm{U}(i)$.

Example 28. Let $\mathcal{M}$ be the $\mathrm{LTS^U}$ from Ex. 7. Recall that $\mathcal{M} \not\models \mathsf{Kh}_j(h, g)$. The conflicting plan is *ebmfsp*, which does not lead to a good cake. Thus, if agent j is able to tell apart *ebmfsp* from *ebfmsp* (which is the good plan), she would be able to know how to bake a good cake, provided she has the ingredients. If agent j *learns* that the order of the actions matters (so *ebmfsp* is now considered distinguishable from *ebfmsp*), the set $\pi = \{ebfmsp, ebmfsp\}$ is split into two singleton sets (i.e., $\mathrm{U}'(j) = \{\{ebfmsp\}, \{ebmfsp\}\}$, where $\mathrm{U}'(j)$ are the indistinguishability sets in the updated model). After the splitting, the agent knows how to achieve g given h.

We define an operation that eliminates uncertainty between specific plans. First, we introduce some notation.

Definition 29. Let $\pi, \pi_1, \pi_2 \in 2^{\mathsf{Act}^*}$, and $S \subseteq 2^{\mathsf{Act}^*}$. We write $\pi = \pi_1 \uplus \pi_2$ iff $\pi = \pi_1 \cup \pi_2$ and $\pi_1 \cap \pi_2 = \emptyset$. For $\pi \in S$ and $\pi = \pi_1 \uplus \pi_2$, define $S^{\pi}_{\{\pi_1,\pi_2\}} \subseteq 2^{\mathsf{Act}^*}$ as the result of splitting, within S, the set of plans π into π_1 and π_2. In other words, $S^{\pi}_{\{\pi_1,\pi_2\}} := (S \setminus \{\pi\}) \cup \{\pi_1, \pi_2\}$.

We define a relation that links sets before an update with those after the update.

Definition 30. Let $S, S' \subseteq 2^{\mathsf{Act}^*}$; and let $\sigma_1, \sigma_2 \in \mathsf{Act}^*$ be such that $\sigma_1 \neq \sigma_2$. We write $S \rightsquigarrow^{\sigma_1}_{\sigma_2} S'$ if and only if either

- $S' = S$ and there is no $\pi \in S$ satisfying $\{\sigma_1, \sigma_2\} \subseteq \pi$, or
- $S' = S^{\pi}_{\{\pi_1,\pi_2\}}$ for some $\pi \in S$ satisfying $\{\sigma_1, \sigma_2\} \subseteq \pi$, with $\pi_1, \pi_2 \in 2^{\mathsf{Act}^*}$ such that $\pi = \pi_1 \uplus \pi_2$ and $\sigma_1 \in \pi_1$, $\sigma_2 \in \pi_2$.

Notice that $\rightsquigarrow^{\sigma_1}_{\sigma_2}$ is serial. Moreover, if S is the set of sets of plans for a given agent i in some $\mathrm{LTS^U}$ (i.e., $S = \mathrm{U}(i)$) and S' is a set satisfying $S \rightsquigarrow^{\sigma_1}_{\sigma_2} S'$, then the structure resulting from replacing S by S' is an $\mathrm{LTS^U}$.

Definition 31. Let $\mathrm{U} = \{\mathrm{U}(i)\}_{i \in \mathsf{Agt}}$ and $\mathrm{U}' = \{\mathrm{U}'(i)\}_{i \in \mathsf{Agt}}$ assign, to every agent in Agt, a non-empty collection of pairwise disjoint non-empty sets of plans (so $\mathrm{U}(i), \mathrm{U}'(i) \subseteq 2^{\mathsf{Act}^*}$); let σ_1, σ_2 be plans in Act^*. We write $\mathrm{U} \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathrm{U}'$ iff $\mathrm{U}(i) \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathrm{U}'(i)$ for every $i \in \mathsf{Agt}$. If $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ is an $\mathrm{LTS^U}$, denote by $\mathcal{M}^{\mathrm{U}}_{\mathrm{U}'}$ the $\mathrm{LTS^U}$ obtained by replacing U by U'.

Definition 32. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ be an $\mathrm{LTS^U}$ and $w \in \mathrm{W}$. For $\sigma_1 \neq \sigma_2$,

$$\mathcal{M}, w \models \langle \sigma_1 \not\sim \sigma_2 \rangle \varphi \quad \mathit{iff}_{def} \text{ there is } \mathrm{U}' \text{ such that } \mathrm{U} \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathrm{U}' \text{ and } \mathcal{M}^{\mathrm{U}}_{\mathrm{U}'}, w \models \varphi.$$

As usual, we define $[\sigma_1 \not\sim \sigma_2]\varphi := \neg\langle \sigma_1 \not\sim \sigma_2 \rangle \neg\varphi$.

Formulas like $\langle \sigma_1 \not\sim \sigma_2 \rangle \varphi$ state that *"after it is (publicly) stated that plans σ_1 and σ_2 are distinguishable, φ holds"*. For instance, in Ex. 28, $\langle ebmfsp \not\sim ebfmsp \rangle \mathsf{Kh}_j(h, g)$ means that *"after it is stated that ebmfsp and ebfmsp are different, agent j knows how to bake a good cake, provided she has the ingredients"*. This modality has some natural properties. First, it is normal and serial but not deterministic (1 through 4 below). Moreover, it represents an idempotent action in which order does not matter and in which two consecutive applications cannot always be collapsed into a single one (5 through 8 below).

Proposition 33. *It follows from the semantics (Def. 32) that:*

1. $\models [\sigma_1 \not\sim \sigma_2](\varphi \to \psi) \to ([\sigma_1 \not\sim \sigma_2]\varphi \to [\sigma_1 \not\sim \sigma_2]\psi)$.
2. *If* $\models \varphi$, *then* $\models [\sigma_1 \not\sim \sigma_2]\varphi$.
3. $\models [\sigma_1 \not\sim \sigma_2]\varphi \to \langle \sigma_1 \not\sim \sigma_2 \rangle\varphi$.
4. $\not\models \langle \sigma_1 \not\sim \sigma_2 \rangle\varphi \to [\sigma_1 \not\sim \sigma_2]\varphi$.
5. $\models [\sigma_1 \not\sim \sigma_2][\sigma_1 \not\sim \sigma_2]\varphi \leftrightarrow [\sigma_1 \not\sim \sigma_2]\varphi$.
6. $\models \langle \sigma_1 \not\sim \sigma_2 \rangle\langle \sigma_3 \not\sim \sigma_4 \rangle\varphi \leftrightarrow \langle \sigma_3 \not\sim \sigma_4 \rangle\langle \sigma_1 \not\sim \sigma_2 \rangle\varphi$.
7. $\models [\sigma_1 \not\sim \sigma_2][\sigma_3 \not\sim \sigma_4]\varphi \leftrightarrow [\sigma_3 \not\sim \sigma_4][\sigma_1 \not\sim \sigma_2]\varphi$.
8. $\mathcal{M}, w \models \langle \sigma_1 \not\sim \sigma_2 \rangle\langle \sigma_3 \not\sim \sigma_4 \rangle\varphi$ *does not imply* $\exists \sigma_5, \sigma_6$ *with* $\mathcal{M}, w \models \langle \sigma_5 \not\sim \sigma_6 \rangle\varphi$.

Proof. Item 1 follows from the $\forall$-pattern in the semantics of $[\sigma_1 \not\sim \sigma_2]$. Item 2 holds because the structure resulting from the operation is an $\mathrm{LTS}^{\mathrm{U}}$. The seriality of $\rightsquigarrow^{\sigma_1}_{\sigma_2}$ ensures Item 3, and its non-determinism (there is more than one way to split a set) ensures Item 4. For Item 5, note that, once the sets including the given plans have been split, the operation does nothing. For Item 6, there are several cases. The one we should focus on is that in which all of the involved plans are in the same set π. Since we are talking about existential modalities, a partition of π into three parts on the left side of the equivalence can always be reproduced by the right part and viceversa. Item 7 can be proved using Item 6. Finally, for Item 8 it is enough to notice that, if the set of plans containing σ_1, σ_2 is different from that containing σ_3, σ_4, then the two actions refine two indistinguishability sets, something that cannot be replicated by a single application. $\square$

The new dynamic modality preserves knowing how information when it refers to propositional formulas (Item 1 below). And it is not trivial, in the sense that it leads to an increase of knowledge in certain situations (Item 2).

Proposition 34. *Let* φ, ψ *be propositional formulas; take* $\sigma_1, \sigma_2 \in \mathsf{Act}^*$ *with* $\sigma_1 \neq \sigma_2$.

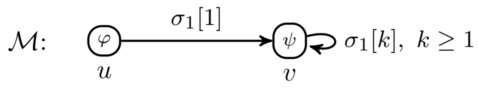
1. $\models \mathsf{Kh}_i(\varphi, \psi) \to [\sigma_1 \not\sim \sigma_2]\mathsf{Kh}_i(\varphi, \psi)$.
2. *If* φ *and* ψ *are satisfiable, then* $\neg\mathsf{Kh}_i(\varphi, \psi) \wedge \langle \sigma_1 \not\sim \sigma_2 \rangle \mathsf{Kh}_i(\varphi, \psi)$ *is satisfiable.*

Proof. For Item 1, suppose $\mathcal{M}, w \models \mathsf{Kh}_i(\varphi, \psi)$. Then there is $\pi \in \mathrm{U}(i)$ s.t. $[\![\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi([\![\varphi]\!]^{\mathcal{M}}) \subseteq [\![\psi]\!]^{\mathcal{M}}$. Let $\sigma_1, \sigma_2 \in \mathsf{Act}^*$. If $\sigma_1 \notin \pi$ or $\sigma_2 \notin \pi$, then π does not change and is still the witness for $\mathsf{Kh}_i(\varphi, \psi)$. If, however, $\sigma_1, \sigma_2 \in \pi$, there will be a partition of π, $\{\pi_1, \pi_2\}$ such that $\mathrm{U}(i) \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathrm{U}(i)^{\pi}_{\{\pi_1, \pi_2\}}$. But this does not cause any problem since $[\![\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi) \subseteq \mathrm{SE}(\pi_k)$ and $\mathrm{R}_{\pi_k}([\![\varphi]\!]^{\mathcal{M}}) \subseteq \mathrm{R}_\pi([\![\varphi]\!]^{\mathcal{M}}) \subseteq [\![\psi]\!]^{\mathcal{M}}$, for $k \in \{1, 2\}$. Here, agent i knew how to go from φ-states to ψ-states via π. Weakening such a π by splitting it into two pieces still works, allowing the agent to choose between π_1 or π_2 as her next witness. Since all the cases for σ_1 and σ_2 are covered, we can conclude that $\mathcal{M}, w \models [\sigma_1 \not\sim \sigma_2]\mathsf{Kh}_i(\varphi, \psi)$.

For Item 2, let $\sigma_1 \neq \sigma_2$, and let $\mathcal{M} = \langle \{u, v\}, \mathrm{R}, \{\mathrm{U}(i)\}, \mathrm{V} \rangle$ be a model such that

- $\mathcal{M}, u \models \varphi$ and $\mathcal{M}, v \models \psi$;
- $\mathrm{R}_{\sigma_1[1]} = \{(u, v), (v, v)\}$ and $\mathrm{R}_{\sigma_1[j]} = \{(v, v)\}$ for all $j = 2, \ldots, |\sigma_1|$;
- $\mathrm{R}_{\sigma_2[k]} = \emptyset$ for some $k = 1, \ldots, |\sigma_2|$ (since $\sigma_1 \neq \sigma_2$); and
- $\mathrm{U}(i) = \{\{\sigma_1, \sigma_2\}\}$.

The model $\mathcal{M}$ is depicted below (the plan σ_2 is omitted):



Since $u \notin \mathrm{SE}(\sigma_2) = \mathrm{SE}(\{\sigma_1, \sigma_2\})$, we have that $\mathcal{M} \models \neg\mathsf{Kh}_i(\varphi, \psi)$. However, $\mathcal{M} \models \langle \sigma_1 \not\sim \sigma_2 \rangle \mathsf{Kh}_i(\varphi, \psi)$ as $\{\{\sigma_1, \sigma_2\}\} \rightsquigarrow^{\sigma_1}_{\sigma_2} \{\{\sigma_1\}, \{\sigma_2\}\}$ and $\{\sigma_1\}$ acts as a witness. $\square$

However, Item 1 in Prop. 34 fails for arbitrary formulas.

Example 35. Consider again Ex. 7, and let $\theta = \mathsf{Kh}_j(h, g)$. Since $\mathcal{M} \not\models \theta$, then we have $\mathcal{M} \models \mathsf{Kh}_j(\theta, \neg\theta)$. Since $\mathcal{M} \models \langle \mathit{ebmfsp} \not\sim \mathit{ebfmsp} \rangle \theta$, it cannot be the case that $\mathcal{M} \models \langle \mathit{ebmfsp} \not\sim \mathit{ebfmsp} \rangle \mathsf{Kh}_j(\theta, \neg\theta)$. Thus, $\mathcal{M} \not\models \mathsf{Kh}_j(\theta, \neg\theta) \to [\mathit{ebmfsp} \not\sim \mathit{ebfmsp}] \mathsf{Kh}_j(\theta, \neg\theta)$, falsifying Item 1 for arbitrary formulas.

The new modality can explicitly refer to plans. Thus, as it could be expected, it adds expressivity w.r.t. the basic knowing how logic.

Proposition 36. $\mathsf{L}_{\mathsf{Ref}}$ *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.

Proof. The single agent $\mathrm{LTS}^{\mathrm{U}}$s models $\mathcal{M}$ and $\mathcal{M}'$ depicted below (with $\mathrm{U}(i) := \{\{a\}\}$ and $\mathrm{U}'(i) := \{\{a, b\}\}$) are $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar; thus, they satisfy the same formulas in $\mathsf{L}_{\mathsf{Kh}_i}$.

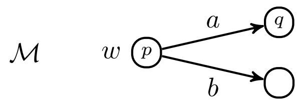
Still, $\mathcal{M}, w \not\models \langle a \not\sim b \rangle \mathsf{Kh}_i(p, q)$ since $\mathrm{U}(i) \rightsquigarrow^{a}_{b} \mathrm{U}(i)$, whereas $\mathcal{M}', w' \models \langle a \not\sim b \rangle \mathsf{Kh}_i(p, q)$, since there is $\mathrm{U}''(i) = \{\{a\}, \{b\}\}$ such that $\mathrm{U}'(i) \rightsquigarrow^{a}_{b} \mathrm{U}''(i)$. $\square$

We conclude by showing that *uniform substitution* fails in $\mathsf{L}_{\mathsf{Ref}}$. Uniform substitution establishes that given a valid formula φ, we can uniformly replace any propositional symbol appearing in φ by an arbitrary formula, and obtain a valid formula. The property fails for many dynamic logics. For example, the logics $\mathsf{PAL}_{\mathsf{Kh}_i}$ and $\mathsf{AUL}_{\mathsf{Kh}_i}$ introduced in the previous section are not closed under uniform substitution. Particular cases where the property fails are the respective RAtom axioms, which are valid for $p \in \mathsf{Prop}$ but validity is not preserved under arbitrary substitutions.

Failure of uniform substitution is not so problematic in the cases of $\mathsf{PAL}_{\mathsf{Kh}_i}$ and $\mathsf{AUL}_{\mathsf{Kh}_i}$ over the class $\mathbf{M_{BA}}$ and, more generally, for dynamic logics with reduction axioms. In these cases, the dynamic modalities can be eliminated to obtain an equivalent formula which is closed under substitutions. The case of $\mathsf{L}_{\mathsf{Ref}}$ is more complex. Its increased expressive power w.r.t. $\mathsf{L}_{\mathsf{Kh}_i}$ implies that reduction axioms cannot be defined. This, in addition to the failure of uniform substitution, indicates that complete axiomatizations might be difficult to find.

Proposition 37. *Uniform substitution fails in* $\mathsf{L}_{\mathsf{Ref}}$.

Proof. The formula $p \to \langle a \not\sim b \rangle p$ is valid for p a propositional symbol, since $\langle a \not\sim b \rangle$ does not change the valuation. Consider the model below, with $\mathrm{U}(i) := \{\{a, b\}\}$:



Consider now the substitution of p by $\neg\mathsf{Kh}_i(p,q)$ in the formula above. The resulting formula is $\neg\mathsf{Kh}_i(p,q) \rightarrow \langle a \nsim b\rangle\neg\mathsf{Kh}_i(p,q)$. It is clear that $\mathcal{M} \models \neg\mathsf{Kh}_i(p,q)$, but $\mathcal{M} \not\models \langle a \nsim b\rangle\neg\mathsf{Kh}_i(p,q)$. Thus, $\neg\mathsf{Kh}_i(p,q) \rightarrow \langle a \nsim b\rangle\neg\mathsf{Kh}_i(p,q)$ is not valid. $\square$

It is well-known that the lack of uniform substitution in many dynamic logics poses a serious challenge in obtaining complete axiomatizations (see, e.g., [41]). This issue has been dealt with in, e.g., [42, 43], for sabotage modal logics. Therein, hybrid logic machinery is used to obtain complete axiomatic systems. This approach seems difficult to apply in our setting, as the language is unable to talk about the actual plans witnessing a formula. We will take a different path in Sec. 5.

4.2 Arbitrary refinement over plans

The 'refinement' relation described by $\langle \sigma_1 \nsim \sigma_2\rangle$ indicates precisely the plans that should be distinguished, with the modality itself quantifying over the different ways of doing so. The operation defined below is more abstract: taking inspiration from other proposals that quantify over epistemic actions (e.g., arbitrary announcements [44], arbitrary arrow updates [45], group announcements [46], coalition announcements [47], arbitrary radical upgrades [48]), it quantifies over the different ways in which the agent's indistinguishability can be refined.

Definition 38. Let $\mathcal{M}$ be an $\mathrm{LTS}^{\mathrm{U}}$ and $w \in \mathrm{D}_{\mathcal{M}}$. Then,

$$\mathcal{M}, w \models \langle \nsim \rangle\varphi \ \textit{iff}_{\textit{def}} \text{ there are } \sigma_1, \sigma_2 \in \mathsf{Act}^* \text{ such that } \mathcal{M}, w \models \langle \sigma_1 \nsim \sigma_2\rangle\varphi.$$

As usual $[\nsim]\varphi = \neg\langle\nsim\rangle\neg\varphi$. We denote $\mathsf{L}_{\mathsf{ARef}}$ (for "arbitrary refinement") as the extension of $\mathsf{L}_{\mathsf{Kh}_i}$ with the modality $\langle\nsim\rangle$.

Let us discuss some ways in which this modality can be used. Knowing how operators are *goal-directed*: the agent looks for a suitable course of action that makes her achieve a certain state. In $\mathsf{L}_{\mathsf{ARef}}$ we can define an operator that indicates whether it is possible for the agent to *learn how* to achieve a given goal: it is enough to ask whether there is a way of spliting *some* existing set of plans so that, afterwards, the agent knows how to achieve φ given ψ. Let L_{Lh} (for "learning how") be $\mathsf{L}_{\mathsf{Kh}_i}$ extended with the ternary modality $\langle\psi,\varphi\rangle_i\chi$ defined as

$$\langle\psi,\varphi\rangle_i\chi := \langle\nsim\rangle(\mathsf{Kh}_i(\psi,\varphi) \wedge \chi) \qquad \text{(for its `dual', } [\psi,\varphi]_i\chi := \neg\langle\psi,\varphi\rangle_i\neg\chi).$$

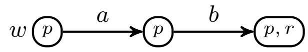
Thus, $\langle\psi,\varphi\rangle_i\chi$ expresses that agent i can learn how to achieve φ given ψ, and that after this learning operation takes place, χ holds. One can further define $\mathsf{L}_i(\psi,\varphi) := \langle\psi,\varphi\rangle_i\top$, an abbreviation for *"the agent i can learn how to make φ true in the presence of ψ"*. This modality, which can be seen as defining a notion of "learnability" in this knowing how context, has the following properties.

Proposition 39. *It follows from the semantics that:*

1. $\not\models \mathsf{L}_i(\varphi,\psi)$*;*
2. $\mathsf{L}_i(\varphi,\psi) \wedge \mathsf{L}_i(\varphi,\neg\psi)$ *is satisfiable.*

Proof. Item 1 shows that not everything is learnable by an agent. The (un)availability of certain actions in an LTS^{U} restricts what can be learnt. Consider the following single-agent LTS^{U} $\mathcal{M}$, with the set $\text{U}(i)$ shown on the right.

$\mathcal{M}$



$\text{U}(i) = \{\ \{ab, a\}, \{\epsilon\}\ \}$

Note that $\mathcal{M}, w \not\models \mathsf{Kh}_i(p, r)$. The set $\{ab, a\}$ is not executable at every p-state, it is only executable at w. On the other hand, $\{\epsilon\}$ is executable everywhere, but does not always lead to r-states. Moreover, $\mathcal{M}, w \not\models \mathsf{L}_i(p, r)$. The set $\{\epsilon\}$ cannot be refined, and no refinement of $\{ab, a\}$ does the work. Therefore, agent i cannot learn how to make r true when p holds.

For Item 2 consider $\mathcal{M}'$ as in Prop. 36. As said, $\mathcal{M}', w' \not\models \mathsf{Kh}_i(p, q)$. However, there is a way to learn how to achieve q given p by splitting the set $\{a, b\}$; hence, $\mathcal{M}', w' \models \mathsf{L}_i(p, q)$ (witness $\{a\}$) but also $\mathcal{M}', w' \models \mathsf{L}_i(p, \neg q)$ (witness $\{b\}$). $\square$

Item 1 in the previous proposition shows how, in certain scenarios, there is no room for learning. For instance, there might be no way to learn how to cure a disease if there is no doctor available. Item 2 shows how the agent might be able to learn not only how to make a formula true under a given condition, but, at the same time, how to make the same formula false under the same condition.

Let us go back to discussing the properties of $\mathsf{L}_{\mathsf{ARef}}$, and their impact in obtaining complete axiomatizations. The modality $\langle \not\sim \rangle$ is normal and serial, and satisfies natural properties of Monotonicity and Weakening. Moreover, it is a non-idempotent action.

Proposition 40. *It follows from the semantics (Def. 38) that:*

1. $\models [\not\sim](\varphi \to \psi) \to ([\not\sim]\varphi \to [\not\sim]\psi)$.
2. *If* $\models \varphi$, *then* $\models [\not\sim]\varphi$.
3. $\models [\not\sim]\varphi \to \langle \not\sim \rangle \varphi$.
4. $\models \langle \not\sim \rangle \varphi \to \langle \not\sim \rangle (\varphi \vee \psi)$ *and* $\models [\not\sim]\varphi \to [\not\sim](\varphi \vee \psi)$ *(Monotonicity).*
5. $\models \langle \not\sim \rangle (\varphi \wedge \psi) \to \langle \not\sim \rangle \varphi$ *and* $\models [\not\sim](\varphi \wedge \psi) \to [\not\sim]\varphi$ *(Weakening).*
6. $\not\models [\not\sim][\not\sim]\varphi \leftrightarrow [\not\sim]\varphi$.

Proof. Items 1 to 3 follow as properties inherited from $\langle \sigma_1 \not\sim \sigma_2 \rangle$. It is also easy to see that Monotonicity and Weakening hold. For Item 6, simply note that a second application might split a further set. $\square$

Knowing how information on propositional formulas is preserved under $[\not\sim]$, while new knowledge can be obtained by a $\langle \not\sim \rangle$ update.

Proposition 41. *Let* φ, ψ *be propositional formulas. Then,*

1. $\models \mathsf{Kh}_i(\varphi, \psi) \to [\not\sim]\mathsf{Kh}_i(\varphi, \psi)$.
2. *If* φ *and* ψ *are satisfiable, then* $\neg\mathsf{Kh}_i(\varphi, \psi) \wedge \langle \not\sim \rangle \mathsf{Kh}_i(\varphi, \psi)$ *are satisfiable.*

Proof. The proofs are similar to those in Prop. 34. $\square$

By definition, $\models \langle \sigma_1 \not\sim \sigma_2 \rangle \varphi \to \langle \not\sim \rangle \varphi$, but characterizing the exact expressivity relation between the two resulting logics requires further developments. In particular, given the mismatch between the two languages ($\mathsf{L}_{\mathsf{Ref}}$ can talk about specific plans, whereas $\mathsf{L}_{\mathsf{ARef}}$ cannot), it does not seem trivial to give a translation from one logic to the other. However, by using the same argument as in Prop. 36, it is easy to show the following:

Proposition 42. $\mathsf{L}_{\mathsf{ARef}}$ *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.

Uniform substitution fails for $\mathsf{L}_{\mathsf{ARef}}$. This, in combination with its high expressivity, makes it difficult to axiomatize.

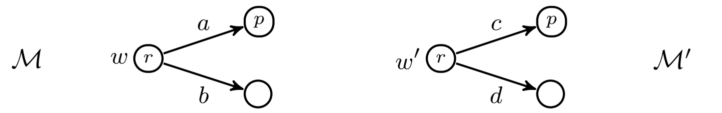
Proposition 43. *Uniform substitution fails in* $\mathsf{L}_{\mathsf{ARef}}$.

Proof. Similar to Prop. 37, and taking $p \to \langle \not\sim \rangle p$ as the original valid formula. $\square$

The problem of defining dynamic operators and corresponding complete axiomatizations will be addressed in the next section. We finish the current one by stating some expressivity connections between the just discussed dynamic modalities.

Proposition 44. *The following propositions are true:*
1. L_{Lh} *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.
2. L_{Lh} *is not more expressive than* $\mathsf{L}_{\mathsf{Ref}}$.

Proof. Item 1 is proved as Prop. 36: the formula $\mathsf{L}_i(p, q)$ distinguishes the two LTS^{U}s. For Item 2 consider the two LTS^{U}s below:



For each model, consider the sets $\text{U}(i) = \{\{a, b\}\}$ and $\text{U}'(i) = \{\{c, d\}\}$. Since L_{Lh} cannot explicitely talk about plans, $\mathcal{M}, w$ and $\mathcal{M}', w'$ are indistinguishable in L_{Lh}. In $\mathsf{L}_{\mathsf{Ref}}$ on the other hand, $\mathcal{M}, w \models \langle a \not\sim b \rangle \mathsf{Kh}_i(r, p)$ and $\mathcal{M}', w' \not\models \langle a \not\sim b \rangle \mathsf{Kh}_i(r, p)$. $\square$

5 Epistemic updates: extending the basic language

A recurring issue when defining both ontic and epistemic updates is the lack of reduction axioms over a general class of models. The reason for this is that the expressivity of the basic language is not enough to describe the effects produced by the update modalities. This makes sense, as the modality Kh_i is very simple, and cannot express properties about explicit courses of action. This section explores a different alternative: enrich the underlying static language $\mathsf{L}_{\mathsf{Kh}_i}$ with a modality to explicitly talk about the actions (or courses of actions) that the agents can take. This modality is simply the basic modal logic operator $[a]$ (with $a \in \mathsf{Act}$).

5.1 The extended basic logic

We start by introducing the syntax and semantics of the logic $\mathsf{L}_{\mathsf{Kh}_i,\Box}$, an extension of $\mathsf{L}_{\mathsf{Kh}_i}$ with the standard $[a]$ modality (see, e.g., [5, 6]).

Definition 45. Formulas of the language $\mathsf{L}_{\mathsf{Kh}_i,\Box}$ are defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi) \mid [a]\varphi,$$

with $p \in \mathsf{Prop}$, $i \in \mathsf{Agt}$ and $a \in \mathsf{Act}$. As usual, we define $\langle a\rangle\varphi$ as $\neg[a]\neg\varphi$.

Axioms	Taut	$\vdash \varphi$ for φ a propositional tautology
	DistA	$\vdash \mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$
	TA	$\vdash \mathsf{A}\varphi \to \varphi$
	$\mathsf{4KhA}$	$\vdash \mathsf{Kh}_i(\psi,\varphi) \to \mathsf{A}\mathsf{Kh}_i(\psi,\varphi)$
	$\mathsf{5KhA}$	$\vdash \neg\mathsf{Kh}_i(\psi,\varphi) \to \mathsf{A}\neg\mathsf{Kh}_i(\psi,\varphi)$
	KhA	$\vdash (\mathsf{A}(\chi \to \psi) \wedge \mathsf{Kh}_i(\psi,\varphi) \wedge \mathsf{A}(\varphi \to \theta)) \to \mathsf{Kh}_i(\chi,\theta)$
	$\mathsf{Dist}\Box$	$\vdash [a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$
	$\mathsf{A}\Box$	$\vdash \mathsf{A}\varphi \to [a]\varphi$
	G	$\vdash \mathsf{Kh}_i(\varphi,\bot) \to \mathsf{Kh}_j(\varphi,\bot)$
Rules	MP	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
	NecA	From $\vdash \varphi$ infer $\vdash \mathsf{A}\varphi$

Table 4: Axiomatization $\mathcal{L}_{\mathsf{Kh}_i,\Box}$ for $\mathsf{L}_{\mathsf{Kh}_i,\Box}$ w.r.t. $\text{LTS}^{\text{U}}s$.

We introduce now the usual semantics for the $[a]$ modality.

Definition 46. Let $\mathcal{M} = \langle \text{W}, \text{R}, \text{U}, \text{V}\rangle$ be an LTS^{U}, $a \in \mathsf{Act}$ and φ be a $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-formula, the semantics of a formula $[a]\varphi$ is defined as

$$\mathcal{M}, w \models [a]\varphi \quad \textit{iff} \quad \mathcal{M}, v \models \varphi \text{ for all } v \in \text{R}_a(w).$$

The axiom system in Table 4, extending that in Table 1 with the well-known axioms $\mathsf{Dist}\Box$ and $\mathsf{A}\Box$, is sound and complete for $\mathsf{L}_{\mathsf{Kh}_i,\Box}$. Soundness easily follows from the validity of the new axioms. For completeness, note first how the necessitation rule for $[a]$ is derivable using NecA, $\mathsf{A}\Box$ and MP. Moreover, no additional 'interaction' axioms are needed: the knowing how modality does not make explicit the plan that serves as a witness. Then, the witnesses for formulas of the form $\mathsf{Kh}_i(\psi,\varphi)$ and $\langle a\rangle\varphi$ can be kept independent from each other. This is used in the completeness proof below (in particular, in the definition of the canonical model), in a way similar to that of [30, 31] for $\mathsf{L}_{\mathsf{Kh}_i}$.

Definition 47. Let $\Gamma \cup \{\varphi\}$ be a set of $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-formulas. The notions of deduction ($\Gamma \vdash \varphi$) and theoremhood ($\vdash \varphi$) are as usual. A set Γ is $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-consistent if $\Gamma \nvdash \bot$, and is maximally $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-consistent (MCS) if it is $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-consistent and for every $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\} \vdash \bot$.

Definition 48. Let Φ be the set of all MCSs in $\mathsf{L}_{\mathsf{Kh}_i,\Box}$. For any $\Delta \in \Phi$, define:

$$\Delta|_{\mathsf{A}} := \{\chi \in \Delta \mid \chi = \mathsf{A}\psi\}, \quad \Delta|_{\mathsf{Kh}_i} := \{\chi \in \Delta \mid \chi = \mathsf{Kh}_i(\psi,\varphi)\}, \quad \Delta|_{\mathsf{Kh}} := \bigcup_{i\in\mathsf{Agt}} \Delta|_{\mathsf{Kh}_i},$$
$$\Delta|_{\neg\mathsf{A}} := \{\chi \in \Delta \mid \chi = \neg\mathsf{A}\psi\}, \ \Delta|_{\neg\mathsf{Kh}_i} := \{\chi \in \Delta \mid \chi = \neg\mathsf{Kh}_i(\psi,\varphi)\}, \ \Delta|_{\neg\mathsf{Kh}} := \bigcup_{i\in\mathsf{Agt}} \Delta|_{\neg\mathsf{Kh}_i}.$$

Let Γ be a set in Φ. Define $\mathsf{Act}_i^\Gamma := \{\langle\psi,\varphi\rangle \mid \mathsf{Kh}_i(\psi,\varphi) \in \Gamma\}$ for every agent $i \in \mathsf{Agt}$, and $\mathsf{Act}^\Gamma := \bigcup_{i\in\mathsf{Agt}} \mathsf{Act}_i^\Gamma \cup \mathsf{Act}$.

Notice that since $\vdash \top$, we have $\mathsf{A}\top = \mathsf{Kh}_i(\neg\top,\bot) \in \Gamma$, for every $i \in \mathsf{Agt}$. Thus, $\mathsf{Act}^\Gamma \neq \emptyset$. Moreover, this set is denumerable since Agt is finite and non-empty, and Act is denumerable. With this property at hand, using a slight variation of what is done in [30, 31], the definition of the canonical model is as follows.

Definition 49. Let $\Gamma \in \Phi$. The structure $\mathcal{M}^\Gamma = \langle \text{W}^\Gamma, \text{R}^\Gamma, \text{U}^\Gamma, \text{V}^\Gamma\rangle$ over $\mathsf{Act}^\Gamma \cup \mathsf{Act}$, Agt and Prop is defined as follows.

- $\text{W}^\Gamma := \{\Delta \in \Phi \mid \Delta|_{\mathsf{A}} = \Gamma|_{\mathsf{A}}\}$.

- $\mathrm{R}^{\Gamma}_{\langle\psi,\varphi\rangle} := \bigcup_{i\in\mathsf{Agt}} \mathrm{R}^{\Gamma}_{\langle\psi,\varphi\rangle^i}$, with
  $\mathrm{R}^{\Gamma}_{\langle\psi,\varphi\rangle^i} = \{(\Delta_1,\Delta_2) \in (\mathrm{W}^{\Gamma})^2 \mid \mathsf{Kh}_i(\psi,\varphi) \in \Gamma, \psi \in \Delta_1, \varphi \in \Delta_2\}$,
- $\mathrm{R}^{\Gamma}_a := \{(\Delta_1,\Delta_2) \in (\mathrm{W}^{\Gamma})^2 \mid \text{for all } \varphi \in \mathsf{L}_{\mathsf{Kh}_i,\Box} : [a]\varphi \in \Delta_1 \text{ implies } \varphi \in \Delta_2\}$,
- $\mathrm{U}^{\Gamma}(i) := \left\{\{\langle\psi,\varphi\rangle\} \mid \langle\psi,\varphi\rangle \in \mathsf{Act}^{\Gamma}_i\right\}$,
- $\mathrm{V}^{\Gamma}(\Delta) := \{p \in \mathsf{Prop} \mid p \in \Delta\}$.

Since the Kh modality does not mention explicitly the plan that serves as a witness, we can define the interpretation of $\mathrm{R}^{\Gamma}_{\langle\psi,\varphi\rangle}$ independently from that of R^{Γ}_a.

Proposition 50. *The structure $\mathcal{M}^{\Gamma} = \langle \mathrm{W}^{\Gamma}, \mathrm{R}^{\Gamma}, \mathrm{U}^{\Gamma}, \mathrm{V}^{\Gamma}\rangle$ is an* $\mathrm{LTS}^{\mathrm{U}}$.

Proof. The set of actions Act^{Γ} is denumerable; thus, $\mathsf{Act}^{\Gamma} \cup \mathsf{Act}$ is also denumerable. Since $\Gamma \in \mathrm{W}^{\Gamma}$, $\mathrm{W}^{\Gamma} \neq \emptyset$. It is enough to show that each $\mathrm{U}^{\Gamma}(i)$ defines a partition over a non-empty subset of $2^{(\mathsf{Act}^{\Gamma})^*}$ since its elements are singletons (thus, mutually disjoint). As we have that $\mathsf{Kh}_i(\bot,\bot) \in \Gamma$, so $\langle\bot,\bot\rangle \in \mathsf{Act}^{\Gamma}_i$ and hence $\{\langle\bot,\bot\rangle\} \in \mathrm{U}^{\Gamma}(i)$; thus, $\mathrm{U}(i) \neq \emptyset$. It is easy to argue that $\emptyset \notin \mathrm{U}^{\Gamma}(i)$. $\square$

The following properties about the canonical model $\mathcal{M}^{\Gamma}$ are useful in what follows (cf. [24, 31] for proofs). Below, we take actions $\langle\psi,\varphi\rangle$ to be those in $\mathsf{Act}^{\Gamma} \setminus \mathsf{Act}$.

Proposition 51. *The following properties hold:*

1. *For $\Delta_1, \Delta_2 \in \mathrm{W}^{\Gamma}$ we have $\Delta_1|_{\mathsf{X}} = \Delta_2|_{\mathsf{X}}$, $\Delta_1|_{\neg\mathsf{X}} = \Delta_2|_{\neg\mathsf{X}}$, $\mathsf{X} \in \{\mathsf{Kh}_i, \mathsf{Kh}, \mathsf{A}\}$.*
2. *Take $\Delta \in \mathrm{W}^{\Gamma}$. If Δ has a $\mathrm{R}^{\Gamma}_{\langle\psi,\varphi\rangle}$-successor, then every $\Delta' \in \mathrm{W}^{\Gamma}$ with $\varphi \in \Delta'$ can be $\mathrm{R}^{\Gamma}_{\langle\psi,\varphi\rangle}$-reached from Δ.*
3. *Let φ be an $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-formula. If $\varphi \in \Delta$ for every $\Delta \in \mathrm{W}^{\Gamma}$, then $\mathsf{A}\varphi \in \Delta$ for every $\Delta \in \mathrm{W}^{\Gamma}$.*
4. *Take ψ, ψ', φ' in $\mathsf{L}_{\mathsf{Kh}_i,\Box}$. Suppose that every $\Delta \in \mathrm{W}^{\Gamma}$ with $\psi \in \Delta$ has a $\mathrm{R}^{\Gamma}_{\langle\psi',\varphi'\rangle}$-successor. Then, $\mathsf{A}(\psi \to \psi') \in \Delta$ for all $\Delta \in \mathrm{W}^{\Gamma}$.*

With these properties at hand, we can prove the truth lemma for $\mathcal{M}^{\Gamma}$.

Lemma 52 (Truth lemma)**.** *Given $\Gamma \in \Phi$, take $\mathcal{M}^{\Gamma} = \langle \mathrm{W}^{\Gamma}, \mathrm{R}^{\Gamma}, \mathrm{U}^{\Gamma}, \mathrm{V}^{\Gamma}\rangle$. Then, for every $\Theta \in \mathrm{W}^{\Gamma}$ and every $\varphi \in \mathsf{L}_{\mathsf{Kh}_i,\Box}$, $\mathcal{M}^{\Gamma}, \Theta \models \varphi$ if and only if $\varphi \in \Theta$.*

Proof. The proof is by induction on φ. We develop here the Kh_i and $[a]$ cases.

Case $[a]\chi$: ($\Rightarrow$) Suppose $\mathcal{M}^{\Gamma}, \Theta \models [a]\chi$. By the definition of $\models$ and IH, for all $\Delta \in \mathrm{R}^{\Gamma}_a(\Theta)$, $\chi \in \Delta$. Suppose $[a]\chi \notin \Theta$. Thus, $\neg[a]\chi = \langle a\rangle\neg\chi \in \Theta$. Let $\Delta^- := \Theta|_{\mathsf{A}} \cup \Theta|_{\neg\mathsf{A}} \cup \{\psi \mid [a]\psi \in \Theta\} \cup \{\neg\chi\}$. Δ^- is consistent. Otherwise there are sets $\{\mathsf{A}\psi_1, \dots, \mathsf{A}\psi_n\} \subseteq \Delta|_{\mathsf{A}}$, $\{\neg\mathsf{A}\psi'_1, \dots, \neg\mathsf{A}\psi'_m\} \subseteq \Delta|_{\neg\mathsf{A}}$ and $\{\psi''_1, \dots, \psi''_l\} \subseteq \{\psi \mid [a]\psi \in \Theta\}$ such that:

$$\vdash \left(\bigwedge_{k=1}^{n} \mathsf{A}\psi_k \wedge \bigwedge_{k=1}^{m} \neg\mathsf{A}\psi'_k \wedge \bigwedge_{k=1}^{l} \psi''_k\right) \to \chi.$$

By NECA, $\mathsf{A}\Box$ and $\mathsf{DIST}\Box$ we have that:

$$\vdash [a]\left(\bigwedge_{k=1}^{n} \mathsf{A}\psi_k \wedge \bigwedge_{k=1}^{m} \neg\mathsf{A}\psi'_k \wedge \bigwedge_{k=1}^{l} \psi''_k\right) \to [a]\chi.$$

Using $\vdash [a](\varphi_1 \wedge \varphi_2) \leftrightarrow ([a]\varphi_1 \wedge [a]\varphi_2)$ we get:

$$\vdash \left(\bigwedge_{k=1}^{n} [a]\mathsf{A}\psi_k \wedge \bigwedge_{k=1}^{m} [a]\neg\mathsf{A}\psi'_k \wedge \bigwedge_{k=1}^{l} [a]\psi''_k \right) \to [a]\chi.$$

By construction, $[a]\psi''_k \in \Theta$. Since $\mathsf{A}\psi_k \in \Theta$, by $\mathsf{4KhA}$ and $\mathsf{A}\Box$, and $[a]\mathsf{A}\psi_k \in \Theta$. With a similar argument with $\mathsf{5KhA}$, $[a]\neg\mathsf{A}\psi'_k \in \Theta$. Hence, $[a]\chi \in \Theta$, a contradiction.

Since Δ^- is consistent, then we have a maximal consistency expansion Δ s.t. $\Delta|_{\mathsf{A}} = \Theta|_{\mathsf{A}}$ (hence, $\Delta \in \mathrm{W}^\Gamma$), $\Delta \in \mathrm{R}^\Gamma_a(\Theta)$ and $\chi \notin \Delta$, a contradiction. Thus, $[a]\chi \in \Theta$.

($\Leftarrow$) Suppose $[a]\chi \in \Theta$. Let $\Delta \in \mathrm{R}^\Gamma_a(\Theta)$, by definition of R^Γ_a, $\chi \in \Delta$. Using IH and the definition of $\models$, for all $\Delta \in \mathrm{R}^\Gamma_a(\Theta)$, we have that $\mathcal{M}^\Gamma, \Delta \models \chi$. Thus, $\mathcal{M}^\Gamma, \Theta \models [a]\chi$.

Case $\mathsf{Kh}_i(\psi, \varphi)$: ($\Rightarrow$) Suppose $\mathcal{M}^\Gamma, \Theta \models \mathsf{Kh}_i(\psi, \varphi)$, and consider two cases.

First suppose $[\![\psi]\!]^{\mathcal{M}^\Gamma} = \emptyset$: Then, each $\Delta \in \mathrm{W}^\Gamma$ is such that $\Delta \notin [\![\psi]\!]^{\mathcal{M}^\Gamma}$, which implies $\psi \notin \Delta$ (by IH) and thus $\neg\psi \in \Delta$ (by maximal consistency). Hence, by Item 3 in Prop. 51, $\mathsf{A}\neg\psi \in \Delta$ for every $\Delta \in \mathrm{W}^\Gamma$. In particular, $\mathsf{A}\neg\psi \in \Theta$ and thus, by KhA and MP ($\mathsf{A}(\psi \to \psi), \mathsf{Kh}(\psi, \bot), \mathsf{A}(\bot \to \varphi) \in \Theta$), $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$.

Now suppose $[\![\psi]\!]^{\mathcal{M}^\Gamma} \neq \emptyset$. From $\mathcal{M}^\Gamma, \Theta \models \mathsf{Kh}_i(\psi, \varphi)$, there is $\{\langle \psi', \varphi' \rangle\} \in \mathrm{U}^\Gamma(i)$ such that $[\![\psi]\!]^{\mathcal{M}^\Gamma} \subseteq \mathrm{SE}(\{\langle \psi', \varphi' \rangle\})$ and $\mathrm{R}^\Gamma_{\{\langle \psi', \varphi' \rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma}) \subseteq [\![\varphi]\!]^{\mathcal{M}^\Gamma}$.

In other words, there is $\langle \psi', \varphi' \rangle \in \mathsf{Act}^\Gamma_i$ such that

1. for all $\Delta \in \mathrm{W}^\Gamma$, if $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$ then $\Delta \in \mathrm{SE}(\{\langle \psi', \varphi' \rangle\})$, so $\Delta \in \mathrm{SE}(\langle \psi', \varphi' \rangle)$ and therefore Δ has a $\mathrm{R}^\Gamma_{\langle \psi', \varphi' \rangle}$-successor;
2. for all $\Delta' \in \mathrm{W}^\Gamma$, if $\Delta' \in \mathrm{R}^\Gamma_{\{\langle \psi', \varphi' \rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma})$ then $\Delta' \in [\![\varphi]\!]^{\mathcal{M}^\Gamma}$.

Now we reason as follows:

1. Take $\Delta \in \mathrm{W}^\Gamma$ with $\psi \in \Delta$. By IH, $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$ and thus, by Item 1, Δ has a $\mathrm{R}^\Gamma_{\langle \psi', \varphi' \rangle}$-successor. Hence, every $\Delta \in \mathrm{W}^\Gamma$ with $\psi \in \Delta$ has such successor; then (Item 4 in Prop. 51), it follows that $\mathsf{A}(\psi \to \psi') \in \Delta$ for every $\Delta \in \mathrm{W}^\Gamma$. In particular, $\mathsf{A}(\psi \to \psi') \in \Theta$.
2. From $\langle \psi', \varphi' \rangle \in \mathsf{Act}^\Gamma_i$ it follows that $\mathsf{Kh}_i(\psi', \varphi') \in \Gamma$. By the definition of W^Γ, $\Theta|_{\mathsf{Kh}} = \Gamma|_{\mathsf{Kh}}$. Hence, $\mathsf{Kh}_i(\psi', \varphi') \in \Theta$.
3. Since $[\![\psi]\!]^{\mathcal{M}^\Gamma} \neq \emptyset$, there is $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$. By Item 1, Δ should have at least one $\mathrm{R}^\Gamma_{\langle \psi', \varphi' \rangle}$-successor. Then, by Item 2 in Prop. 51, every $\Delta' \in \mathrm{W}^\Gamma$ satisfying $\varphi' \in \Delta'$ can be $\mathrm{R}^\Gamma_{\langle \psi', \varphi' \rangle}$-reached from Δ; in other words, every $\Delta' \in \mathrm{W}^\Gamma$ satisfying $\varphi' \in \Delta'$ is in $\mathrm{R}^\Gamma_{\langle \psi', \varphi' \rangle}(\Delta)$. Since $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$, every $\Delta' \in \mathrm{W}^\Gamma$ satisfying $\varphi' \in \Delta'$ is in $\mathrm{R}^\Gamma_{\langle \psi', \varphi' \rangle}([\![\psi]\!]^{\mathcal{M}^\Gamma})$. Then, by Item 2, every $\Delta' \in \mathrm{W}^\Gamma$ satisfying $\varphi' \in \Delta'$ is in $[\![\varphi]\!]^{\mathcal{M}^\Gamma}$. By IH on the latter part, every $\Delta' \in \mathrm{W}^\Gamma$ satisfying $\varphi' \in \Delta'$ is such that $\varphi \in \Delta'$. Thus, $\varphi' \to \varphi \in \Delta'$ for every $\Delta' \in \mathrm{W}^\Gamma$. Hence, by Item 3 in Prop. 51, $\mathsf{A}(\varphi' \to \varphi) \in \Delta'$ for every $\Delta' \in \mathrm{W}^\Gamma$. In particular, $\mathsf{A}(\varphi' \to \varphi) \in \Theta$.

Thus, $\{\mathsf{A}(\psi \to \psi'), \mathsf{Kh}_i(\psi', \varphi'), \mathsf{A}(\varphi' \to \varphi)\} \subset \Theta$; and by KhA and MP, $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$.

($\Leftarrow$) Suppose $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$. Thus (Item 1 in Prop. 51), $\mathsf{Kh}_i(\psi, \varphi) \in \Gamma$, so $\langle \psi, \varphi \rangle \in \mathsf{Act}^\Gamma_i$ and therefore $\{\langle \psi, \varphi \rangle\} \in \mathrm{U}^\Gamma(i)$. The rest of the proof is split in two cases.

Assume there is no $\Delta_\psi \in \mathrm{W}^\Gamma$ with $\psi \in \Delta$. Then, by IH, there is no $\Delta_\psi \in \mathrm{W}^\Gamma$ with $\Delta_\psi \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$, i.e., $[\![\neg\psi]\!]^{\mathcal{M}^\Gamma} = \mathrm{W}^\Gamma$. Since $\mathcal{M}^\Gamma$ is an LTS^U (Prop. 50), $\mathcal{M}^\Gamma, \Delta \models \mathsf{Kh}_i(\psi, \chi)$ for any $i \in \mathsf{Agt}$, $\chi \in \mathsf{L}_{\mathsf{Kh}_i,\Box}$ and $\Delta \in \mathrm{W}^\Gamma$; hence, $\mathcal{M}^\Gamma, \Theta \models \mathsf{Kh}_i(\psi, \varphi)$.

Now, assume there is $\Delta_\psi \in \mathrm{W}^\Gamma$ with $\psi \in \Delta_\psi$. We show that the set of plans $\{\langle\psi, \varphi\rangle\} \in \mathrm{U}^\Gamma(i)$ satisfies the requirements. Take $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$. By IH, $\psi \in \Delta$. Moreover, from $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$ and Item 1 in in Prop. 51 it follows that $\mathsf{Kh}_i(\psi, \varphi) \in \Delta$. Then, from $\mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle^i}$'s definition, every $\Delta' \in \mathrm{W}^\Gamma$ with $\varphi \in \Delta'$ is such that $(\Delta, \Delta') \in \mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle^i}$, and therefore such that $(\Delta, \Delta') \in \mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle}$. Now note how, since there is $\Delta_\psi \in \mathrm{W}^\Gamma$ with $\psi \in \Delta_\psi$, there should be $\Delta_\varphi \in \mathrm{W}^\Gamma$ with $\varphi \in \Delta_\varphi$. Suppose otherwise, i.e., suppose there is no $\Delta'' \in \mathrm{W}^\Gamma$ with $\varphi \in \Delta''$. Then, $\neg\varphi \in \Delta''$ for every $\Delta'' \in \mathrm{W}^\Gamma$, and hence (Item 3 in Prop. 51) $\mathsf{A}\neg\varphi \in \Delta''$ for every $\Delta'' \in \mathrm{W}^\Gamma$. In particular, $\mathsf{A}\neg\varphi \in \Delta_\psi$. Moreover, from $\mathsf{Kh}_i(\psi, \varphi) \in \Theta$ and Item 1 in Prop. 51, it follows that $\mathsf{Kh}_i(\psi, \varphi) \in \Delta_\psi$. Then, KhE (written as $\mathsf{Kh}_i(\psi, \varphi) \to (\mathsf{A}\neg\varphi \to \mathsf{A}\neg\psi)$) and MP yield $\mathsf{A}\neg\psi \in \Delta_\psi$, and thus (axiom TA) $\neg\psi \in \Delta_\psi$. Hence, $\{\psi, \neg\psi\} \subset \Delta_\psi$, contradicting Δ_ψ's consistency.

The existence of $\Delta_\varphi \in \mathrm{W}^\Gamma$ with $\varphi \in \Delta_\varphi$ implies that $(\Delta, \Delta_\varphi) \in \mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle}$ and thus, since $\langle\psi, \varphi\rangle$ is a basic action, $\Delta \in \mathrm{SE}(\langle\psi, \varphi\rangle)$, and so $\Delta \in \mathrm{SE}(\{\langle\psi, \varphi\rangle\})$. Since Δ is an arbitrary state in $[\![\psi]\!]^{\mathcal{M}^\Gamma}$, $[\![\psi]\!]^{\mathcal{M}^\Gamma} \subseteq \mathrm{SE}(\{\langle\psi, \varphi\rangle\})$ as needed.

Now, take $\Delta' \in \mathrm{R}^\Gamma_{\{\langle\psi,\varphi\rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma})$. Then, there is $\Delta \in [\![\psi]\!]^{\mathcal{M}^\Gamma}$ such that $(\Delta, \Delta') \in \mathrm{R}^\Gamma_{\langle\psi,\varphi\rangle}$. By definition of R^Γ, $\varphi \in \Delta'$ and, by IH, $\Delta' \in [\![\varphi]\!]^{\mathcal{M}^\Gamma}$. Since Δ' is an arbitrary state in $\mathrm{R}^\Gamma_{\{\langle\psi,\varphi\rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma})$, we have $\mathrm{R}^\Gamma_{\{\langle\psi,\varphi\rangle\}}([\![\psi]\!]^{\mathcal{M}^\Gamma}) \subseteq [\![\varphi]\!]^{\mathcal{M}^\Gamma}$ as needed. □

Completeness now follows from the Truth Lemma in the usual way.

Theorem 53. *The axiom system $\mathcal{L}_{\mathsf{Kh}_i,\Box}$ is sound and strongly complete with respect to the class of all models.*

It is easy to see that $\mathsf{L}_{\mathsf{Kh}_i,\Box}$ is strictly more expressive than $\mathsf{L}_{\mathsf{Kh}_i}$. Consider for instance, the models $\mathcal{M}$ and $\mathcal{M}'$ from the proof of Prop. 17. It has been established therein that the models are $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar. However, $\mathcal{M}', w' \models \langle b\rangle\top$ while $\mathcal{M}, w \not\models \langle b\rangle\top$. Thus, they are distinguishable by an $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-formula. Moreover, $\mathsf{L}_{\mathsf{Kh}_i,\Box}$ is also strictly more expressive than the standard multi-modal language $\mathsf{L}_\Box$, as the global modality is definable in the former, but not in the latter.

Now we turn to the decidability of the satisfiability problem of $\mathsf{L}_{\mathsf{Kh}_i,\Box}$. We will use a *filtration* technique (see, e.g., [5]). The filtration we will define generalizes the one introduced in [31] for $\mathsf{L}_{\mathsf{Kh}_i}$, by also handling the basic modal logic modality $[a]$. Moreover, in [31] it is not shown that a filtration always exists for a model and a set of formulas, which is a missing ingredient in the proof. This aspect is fixed here too.

Definition 54 (Σ-equivalence)**.** Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V}\rangle$ be an LTS^U and let Σ be a set of $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-formulas closed under subformulas. Define the equivalence relations $\leftrightsquigarrow_\Sigma \subseteq \mathrm{W} \times \mathrm{W}$ and $\leftrightarrows_\Sigma \subseteq \mathrm{U}_{\mathsf{Agt}} \times \mathrm{U}_{\mathsf{Agt}}$ (with $\mathrm{U}_{\mathsf{Agt}} := \bigcup_{i\in\mathsf{Agt}} \mathrm{U}(i)$) as:

$w \leftrightsquigarrow_\Sigma v$ *iff*$_{def}$ for all $\psi \in \Sigma, \mathcal{M}, w \models \psi$ iff $\mathcal{M}, v \models \psi$,
$\pi \leftrightarrows_\Sigma \pi'$ *iff*$_{def}$ for all $i \in \mathsf{Agt}$ and $\mathsf{Kh}_i(\psi, \varphi) \in \Sigma, \pi$ is a witness for $\mathsf{Kh}_i(\psi, \varphi)$ in $\mathcal{M}$ iff π' is a witness for $\mathsf{Kh}_i(\psi, \varphi)$ in $\mathcal{M}$.

Given $w \in \mathrm{W}$ and $\pi \in \mathrm{U}_{\mathsf{Agt}}$, we define their Σ-equivalence class as $[w]_\Sigma := \{v \in \mathrm{W} \mid w \leftrightsquigarrow_\Sigma v\}$ and $[\pi]_\Sigma := \{\pi' \in 2^{\mathsf{Act}^*} \mid \pi \leftrightarrows_\Sigma \pi'\}$. Although the notation $[\_]_\Sigma$ is overloaded, its argument will always disambiguate its use.

Definition 55. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V}\rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$ defined over a denumerable set of actions Act, and let Σ be a set of $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-formulas closed under subformulas. Define

$$\begin{aligned}
\mathsf{Act}_i^\Sigma &:= \{a_{[\pi]_\Sigma} \mid \pi \in \mathrm{U}(i) \text{ is a witness of some } \mathsf{Kh}_i(\psi,\varphi) \in \Sigma \text{ in } \mathcal{M}\},\\
\mathsf{Act}_\Box^\Sigma &:= \{a \in \mathsf{Act} \mid [a]\varphi \in \Sigma\}, \text{ and}\\
\mathsf{Act}^\Sigma &:= \textstyle\bigcup_{i\in\mathsf{Agt}} \mathsf{Act}_i^\Sigma \cup \mathsf{Act}_\Box^\Sigma \cup \{a_\emptyset\}.
\end{aligned}$$

The idea behind the definition of Act_i^Σ is that, for each $\mathsf{Kh}_i(\psi,\varphi) \in \Sigma$ that is true at $\mathcal{M}$, we consider an action mimicking the behavior of those sets of plans π that witness the satisfiability of $\mathsf{Kh}_i(\psi,\varphi)$ in $\mathcal{M}$. In turn, $\mathsf{Act}_\Box^\Sigma$ takes into account those action symbols that are explicitly used in formulas of Σ. Finally, $a_\emptyset$ will be used to guarantee the existence of an action in the set Act^Σ, as we will see without provoking any side effect. Now, we are in position of defining the notion of filtration.

Definition 56 (Filtration of $\mathcal{M}$ through Σ)**.** Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V}\rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$ defined over Act; let Σ be a set of $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-formulas that is closed under subformulas. An $\mathrm{LTS}^{\mathrm{U}}$ defined over Act^Σ, $\mathcal{M}^f = \langle \mathrm{W}^f, \mathrm{R}^f, \mathrm{U}^f, \mathrm{V}^f\rangle$, is *a filtration of $\mathcal{M}$ through Σ* if and only if it satisfies the following conditions:

1. $\mathrm{W}^f := \{[w]_\Sigma \mid w \in \mathrm{W}\}$;
2. $\mathrm{V}^f([w]_\Sigma) := \{p \in \Sigma \mid \mathcal{M}, w \models p\}$;
3. $\mathrm{U}(i)^f := \{\{a_{[\pi]_\Sigma}\} \mid a_{[\pi]_\Sigma} \in \mathsf{Act}_i^\Sigma\} \cup \{\{a_\emptyset\}\}$ for each $i \in \mathsf{Agt}$;
4. $\mathrm{R}^f_{a_\emptyset} = \emptyset$ and for all $a_{[\pi]_\Sigma} \in \mathsf{Act}_i^\Sigma$, there exists a non empty set $G_{[\pi]_\Sigma} \subseteq [\pi]_\Sigma$ such that $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_{a_{[\pi]_\Sigma}}$ iff
   - for all $w' \in [w]_\Sigma$ and $\pi' \in G_{[\pi]_\Sigma}$, we have $w' \in \mathrm{SE}(\pi')$, and
   - there are $w'' \in [w]_\Sigma$, $v'' \in [v]_\Sigma$ and $\pi'' \in G_{[\pi]_\Sigma}$ such that $(w'', v'') \in \mathrm{R}_{\pi''}$;
5. for $a \in \mathsf{Act}_\Box^\Sigma$, if $(w, v) \in \mathrm{R}_a$, then $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_a$; and
6. for $a \in \mathsf{Act}_\Box^\Sigma$, if $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_a$ then for all $[a]\varphi \in \Sigma$, $\mathcal{M}, w \models [a]\varphi$ implies $\mathcal{M}, v \models \varphi$.

Def. 56 deserves further comments. The filtration is defined similarly as for the basic modal logic (see, e.g., [5]). The most significant difference is the addition of a denumerable set of actions and relations (Item 4) since we use the set Act^Σ as the set of action names. As a consequence, the actions an agent i considers in $\mathrm{U}^f(i)$ are separated from the actions defined in Act. Moreover, if Σ is finite, the relation $\leftrightarrows_\Sigma$ enables us to obtain a finite set of witnesses for the formulas $\mathsf{Kh}_i(\psi,\varphi) \in \Sigma$, from which (together with $\leftrightsquigarrow_\Sigma$) we also get that Act^Σ and W^f are finite. $\mathrm{U}^f(i)$ is well-defined since it is made only of singletons. Specially, if there is no $\pi \in \mathrm{U}(i)$ such that π is a witness for some $\mathsf{Kh}_i(\varphi,\psi) \in \Sigma$, $\mathrm{U}^f(i) \neq \emptyset$ still holds. This is due to the addition of a vacuous action, $a_\emptyset$, which is not strongly executable in any state. The conditions 5 and 6 are the usual ones for the basic modal logic. In fact, notice that they are not in conflict with the new conditions, as the actions used to talk about formulas with Kh are disjoint from those to talk about formulas with $[a]$ modalities. So, new actions $a_{[\pi]_\Sigma}$ are introduced as a result of the witness for a knowing how, but as they are fresh actions, they do not impact on any formula from Σ. Finally, Item 4 can be rewritten as $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_{a_{[\pi]_\Sigma}}$ if and only if the following conditions hold:

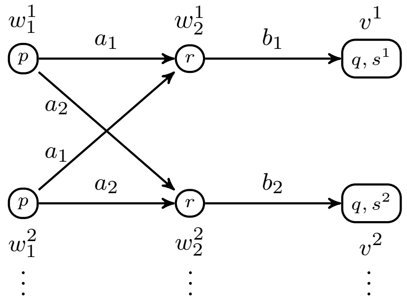
- $[w]_\Sigma \subseteq \bigcap_{\pi' \in G_{[\pi]_\Sigma}} \mathrm{SE}(\pi')$, and
- $(w'', v'') \in \bigcup_{\pi'' \in G_{[\pi]_\Sigma}} \mathrm{R}_{\pi''}$ for some $w'' \in [w]_\Sigma$ and $v'' \in [v]_\Sigma$.

Moreover, $[w]_\Sigma \in \mathrm{SE}(a_{[\pi]_\Sigma})$ iff $[w]_\Sigma \subseteq \bigcap_{\pi' \in G_{[\pi]_\Sigma}} \mathrm{SE}(\pi')$. The example below shows how filtrations preserve the information for evaluating formulas from the input set.

Example 57. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V} \rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$ defined over $\mathsf{Agt} = \{k, j\}$ and $\mathsf{Act} = \{a_i, b_i \mid i \in \mathbb{N}\}$ as:

1. $\mathrm{W} = \{w_1^i \mid i \in \mathbb{N}\} \cup \{w_2^i \mid i \in \mathbb{N}\} \cup \{v^i \mid i \in \mathbb{N}\}$;
2. $\mathrm{R}_{a_i} = \{(w_1^i, w_2^j) \mid j \in \mathbb{N}\}$, $\mathrm{R}_{b_i} = \{(w_2^i, v^i)\}$;
3. $\mathrm{U}(k) = \{\pi_1, \pi_2, \pi_3\}$, where $\pi_1 = \{a_1 b_1\}$, $\pi_2 = \{a_1\}$, $\pi_3 = \{a_i b_i \mid i \in \mathbb{N} \setminus \{1\}\}$, and $\mathrm{U}(j) = \{\{a_i\} \mid i \in \mathbb{N} \setminus \{1\}\}$;
4. $\mathrm{V}(w_1^i) = \{p\}$, $\mathrm{V}(w_2^i) = \{r\}$, $\mathrm{V}(v^i) = \{q, s^i\}$, for all $i \in \mathbb{N}$.

The model just defined is depicted below:



Let $\Sigma = \{\mathsf{Kh}_k(p, q), \mathsf{Kh}_k(p, r), \neg\mathsf{Kh}_j(p, q), \mathsf{Kh}_j(p, r), p, q, r, [b_1]s^1, s^1\}$ a set of formulas closed by subformulas. The respective equivalence classes for the plans in $\mathrm{U}_{\mathsf{Agt}}$ are defined as:

- $[\pi_1]_\Sigma = [\pi_3]_\Sigma = \{\pi_1, \pi_3\}$,
- $[\pi_2]_\Sigma = [\{a_1\}]_\Sigma = \{\{a_1\}\}$, and
- $[\{a_2\}]_\Sigma = [\pi_4]_\Sigma = \{\{a_i\} \mid i \in \mathbb{N} \setminus \{1\}\}$.

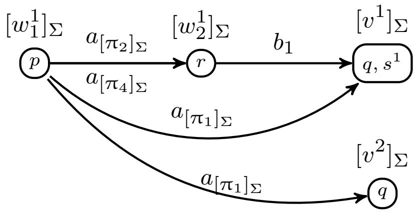
From now on, $\{a_2\}$ will be denoted as π_4 as we did above. Notice that $[\pi_1]_\Sigma$ encompasses the set of plans that are witnesses for $\mathsf{Kh}_k(p, q)$, while $[\pi_2]_\Sigma$ encompasses the witnesses for $\mathsf{Kh}_k(p, r)$. For the case of $\mathsf{Kh}_j(p, q)$ there is no witness under consideration, as the formula does not hold at $\mathcal{M}$. Finally, $[\pi_4]_\Sigma$ encompassess the witnesses for $\mathsf{Kh}_j(p, r)$. Thus, we have $\mathsf{Act}^\Sigma = \{a_\emptyset, a_{[\pi_1]_\Sigma}, a_{[\pi_2]_\Sigma}, a_{[\pi_4]_\Sigma}, b_1\}$.

A possible filtration would be $\mathcal{M}^f = \langle \mathrm{W}^f, \mathrm{R}^f, \mathrm{U}^f, \mathrm{V}^f \rangle$, with:

1. $\mathrm{W}^f := \{[w_1^1]_\Sigma, [w_2^1]_\Sigma, [v^1]_\Sigma, [v^2]_\Sigma\}$, where:
   - $[w_1^1]_\Sigma = \{w_1^i \mid i \in \mathbb{N}\}$,
   - $[w_2^1]_\Sigma = \{w_2^i \mid i \in \mathbb{N}\}$,
   - $[v^1]_\Sigma = \{v^1\}$, and
   - $[v^2]_\Sigma = \{v^i \mid i \in \mathbb{N} \setminus \{1\}\}$.
2. $\mathrm{V}^f([w_1^1]_\Sigma) = \{p\}$, $\mathrm{V}([w_2^1]_\Sigma) = \{r\}$, $\mathrm{V}([v^1]_\Sigma) = \{q, s^1\}$ and $\mathrm{V}([v^2]_\Sigma) = \{q\}$.
3. $\mathrm{U}^f(k) = \{\{a_{[\pi_1]_\Sigma}\}, \{a_{[\pi_2]_\Sigma}\}, \{a_\emptyset\}\}$, and $\mathrm{U}^f(j) = \{\{a_{[\pi_4]_\Sigma}\}, \{a_\emptyset\}\}$;
4. R^f is defined as follows for each $a \in \mathsf{Act}^\Sigma$:

- $R^f_{a_\emptyset} = \emptyset$,
- $R^f_{a_{[\pi_1]_\Sigma}} = \{([w_1^1]_\Sigma, [v^i]_\Sigma) \mid i \in \mathbb{N}\}$ (if $G_{[\pi_1]_\Sigma} = [\pi_1]_\Sigma$),
- $R^f_{a_{[\pi_2]_\Sigma}} = \{([w_1^1]_\Sigma, [w_2^1]_\Sigma)\}$ (if $G_{[\pi_2]_\Sigma} = [\pi_2]_\Sigma$),
- $R^f_{a_{[\pi_4]_\Sigma}} = \{([w_1^1]_\Sigma, [w_2^1]_\Sigma)\}$ (if $G_{[\pi_4]_\Sigma} = [\pi_4]_\Sigma$), and
- $R^f_{b_1} = \{([w_2^1]_\Sigma, [v^1]_\Sigma)\}$.

The filtration $\mathcal{M}^f$ is depicted below:



If, however, we consider $G_{[\pi_1]_\Sigma} = \{\pi_1\}$, then $R^f_{a_{[\pi_1]_\Sigma}} = \{([w_1^1]_\Sigma, [v^1]_\Sigma)\}$, as (w_1^1, v^1) is the only edge that holds the two conditions. Thus, we can observe how there are at least two alternatives for defining a filtration from $\mathcal{M}$ and Σ.

Now we prove that a model and its filtration via Σ satisfy the same formulas in Σ.

Theorem 58. *Let $\mathcal{M} = \langle W, R, U, V\rangle$ be an LTS^U and let Σ be a set of $\mathsf{L}_{\mathsf{Kh}_i,\Box}$-formulas that is closed under subformulas. Then, for all $\psi \in \Sigma$ and $w \in W$, $\mathcal{M}, w \models \psi$ iff $\mathcal{M}^f, [w]_\Sigma \models \psi$. Moreover, if Σ is finite then $\mathcal{M}^f$ is a finite model.*

Proof. We will prove the cases for Kh_i and $[a]$ (Boolean cases are as expected).

Case $\mathsf{Kh}_i(\psi, \varphi)$: ($\Rightarrow$) Suppose $\mathcal{M} \models \mathsf{Kh}_i(\psi, \varphi)$, and let $\pi \in U(i)$ be one of its witnesses. By Def. 55, $a_{[\pi]_\Sigma} \in \mathsf{Act}_i^\Sigma$ and thus, $\{a_{[\pi]_\Sigma}\} \in U^f(i)$. It is enough to prove that this is a witness for $\mathsf{Kh}_i(\psi, \varphi)$ in $\mathcal{M}^f$. By Item 4 of Def. 56, for all $\pi' \in G_{[\pi]_\Sigma}$, $\pi \leftrightarrows_\Sigma \pi'$. Then, for all $\pi' \in G_{[\pi]_\Sigma}$, $[\![\psi]\!]^{\mathcal{M}} \subseteq SE(\pi')$ and $R_{\pi'}([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}}$.

1. Let $[w]_\Sigma \in [\![\psi]\!]^{\mathcal{M}^f}$. By IH, for all $w' \in [w]_\Sigma = [w']_\Sigma$, $w' \in [\![\psi]\!]^{\mathcal{M}}$, thus $[w]_\Sigma \subseteq SE(\pi')$, for all $\pi' \in G_{[\pi]_\Sigma}$. Then, $[w]_\Sigma \in SE(a_{[\pi]_\Sigma})$. Hence, $[\![\psi]\!]^{\mathcal{M}^f} \subseteq SE(\{a_{[\pi]_\Sigma}\})$.
2. Let $([w]_\Sigma, [v]_\Sigma) \in R^f_{a_{[\pi]_\Sigma}}$ be such that $[w]_\Sigma \in [\![\psi]\!]^{\mathcal{M}^f}$. Then, for some $w' \in [w]_\Sigma = [w']_\Sigma$, $v' \in [v]_\Sigma = [v']_\Sigma$ and $\pi' \in G_{[\pi]_\Sigma}$ we have $(w', v') \in R_{\pi'}$. By inductive hypothesis, $w' \in [\![\psi]\!]^{\mathcal{M}}$ and thus $v' \in [\![\varphi]\!]^{\mathcal{M}}$. Again, by inductive hypothesis, $[v']_\Sigma = [v]_\Sigma \in [\![\varphi]\!]^{\mathcal{M}^f}$. Then, $R^f_{\{a_{[\pi]_\Sigma}\}}([\![\psi]\!]^{\mathcal{M}^f}) \subseteq [\![\varphi]\!]^{\mathcal{M}^f}$.

As a consequence, we get $\mathcal{M}^f \models \mathsf{Kh}_i(\psi, \varphi)$.

($\Leftarrow$) Suppose $\mathcal{M}^f \models \mathsf{Kh}_i(\psi, \varphi)$, and let $\pi \in U^f(i)$ be one of its witnesses. If $\pi = \{a_\emptyset\}$, then $[\![\psi]\!]^{\mathcal{M}^f} \subseteq SE(\{a_\emptyset\}) = \emptyset$. By inductive hypothesis, $[\![\psi]\!]^{\mathcal{M}} = \emptyset$ and any $\pi \in U(i)$ is a witness in $\mathcal{M}$ for $\mathsf{Kh}_i(\psi, \varphi)$. If $\pi = \{a_{[\pi']_\Sigma}\}$, then $[\![\psi]\!]^{\mathcal{M}^f} \subseteq SE(\{a_{[\pi']_\Sigma}\})$ and $R^f_{\{a_{[\pi']_\Sigma}\}}([\![\psi]\!]^{\mathcal{M}^f}) \subseteq [\![\varphi]\!]^{\mathcal{M}^f}$. We prove that any $\pi'' \in G_{[\pi']_\Sigma}$ does the job.

1. Let $w \in [\![\psi]\!]^{\mathcal{M}}$, by inductive hypothesis, $[w]_\Sigma \in [\![\psi]\!]^{\mathcal{M}^f}$ and consequently $[w]_\Sigma \in SE(\{a_{[\pi']_\Sigma}\})$. By Item 4 in Def. 56, we have that $[w]_\Sigma \subseteq \bigcap_{\pi'' \in G_{[\pi']_\Sigma}} SE(\pi'')$, and naturally that $w \in \bigcap_{\pi'' \in G_{[\pi']_\Sigma}} SE(\pi'')$. Thus, $[\![\psi]\!]^{\mathcal{M}} \subseteq SE(\pi'')$, for all $\pi'' \in G_{[\pi']_\Sigma}$.

2. Take $(w, v) \in \mathrm{R}_{\pi''}$ with $w \in [\![\psi]\!]^{\mathcal{M}}$. Since $w \in \bigcap_{\pi'' \in G_{[\pi']_\Sigma}} \mathrm{SE}(\pi'')$, necessarily we have $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_{a_{[\pi']_\Sigma}}$. By IH, $[w]_\Sigma \in [\![\psi]\!]^{\mathcal{M}^f}$ and with this $[v]_\Sigma \in [\![\varphi]\!]^{\mathcal{M}^f}$. Again, by IH, $v \in [\![\varphi]\!]^{\mathcal{M}}$. Then, $\mathrm{R}_{\pi''}([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}}$, for any $\pi'' \in G_{[\pi']_\Sigma}$.

Therefore, $\mathcal{M} \models \mathsf{Kh}_i(\psi, \varphi)$.

Case $[a]\varphi$: ($\Rightarrow$) Suppose $\mathcal{M}, w \models [a]\varphi$. Let $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_a$, since $[a]\varphi \in \Sigma$, then $\mathcal{M}, v \models \varphi$. By IH, $\mathcal{M}^f, [v]_\Sigma \models \varphi$. Hence, $\mathcal{M}^f, [w]_\Sigma \models [a]\varphi$.

($\Leftarrow$) Suppose $\mathcal{M}^f, [w]_\Sigma \models [a]\varphi$. Let $(w, v) \in \mathrm{R}_a$, then $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_a$ and $\mathcal{M}^f, [v]_\Sigma \models \varphi$. By IH, $\mathcal{M}, v \models \varphi$. Hence, $\mathcal{M}, w \models [a]\varphi$.

It remains to show that if Σ is finite, then $\mathcal{M}^f$ is finite. First, note that the number of elements in W^f is at most exponential in the number of formulas in Σ. Thus, W^f and V^f are finite. Similarly, as the number of $\mathsf{Kh}_i(\psi, \varphi)$ and $[a]\varphi$ in Σ are finite, Act^Σ is also finite. Thus, R^f and U^f are both finite. □

Example 59. Consider Ex. 57. Notice that the filtration preserves the satisfiability of formulas in Σ. For instance, $\mathcal{M} \models \mathsf{Kh}_k(p, q)$ with π_1 as witness, while $\mathcal{M}^f \models \mathsf{Kh}_k(p, q)$ with $\{a_{[\pi_1]_\Sigma}\}$ as witness. Also, it is clear that both $\mathcal{M} \not\models \mathsf{Kh}_j(p, q)$ and $\mathcal{M}^f \not\models \mathsf{Kh}_j(p, q)$, and that $\mathcal{M}, w^1_2 \models [b_1]s^1$ and $\mathcal{M}^f, [w^1_2]_\Sigma \models [b_1]s^1$. One can verify that this is also the case for other alternative filtrations, as established by Thm. 58.

Considering Thm. 58, it remains to prove that, given a model $\mathcal{M}$ and a set of formulas Σ closed under subformulas, the set of filtrations of $\mathcal{M}$ via Σ is non empty. This can be achieved by defining $G_{[\pi]_\Sigma} = [\pi]_\Sigma \neq \emptyset$ for each $a_{[\pi]_\Sigma} \in \mathsf{Act}^\Sigma_i$, and considering at least one of the two alternatives to define R^f_a for each $a \in \mathsf{Act}^\Sigma_\Box$ given in [5, 6]:

- $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_a$ iff there exists $w' \in [w]_\Sigma$, $v' \in [v]_\Sigma$ such that $(w', v') \in \mathrm{R}_a$; or
- $([w]_\Sigma, [v]_\Sigma) \in \mathrm{R}^f_a$ iff for all $[a]\varphi \in \Sigma$, $\mathcal{M}, w \models [a]\varphi$ then $\mathcal{M}, v \models \varphi$.

The different alternatives for $G_{[\pi]_\Sigma}$ and R^f_a enable us to obtain more than one filtration, all of them satisfying the intended properties.

As we established that every satisfiable formula of $\mathsf{L}_{\mathsf{Kh}_i,\Box}$, is satisfiable in a finite, bounded model, the satisfiability problem for $\mathsf{L}_{\mathsf{Kh}_i,\Box}$ is decidable.

Corollary 60. *The satisfiability problem for* $\mathsf{L}_{\mathsf{Kh}_i,\Box}$ *is decidable.*

5.2 Plan refinement

We defined $\mathsf{L}_{\mathsf{Kh}_i,\Box}$ to obtain enough expressivity to axiomatize some dynamic modalities. Here, we define the modality $[!\sigma]$, inspired by the refinement modality from Sec. 4.1. A formula $[!\sigma]\varphi$ publicly establishes that the plan σ must be distinguished from any other, and in such a context φ holds. It is natural to come up with situations in which this modality can be used. As an example, suppose an agent considers two plans σ_1 and σ_2 as indistinguishable possibilities to reach a location, but a map app detects a strike in path σ_2. Thus, independently of being a "good plan" or not, σ_1 becomes the only choice to be followed.

Definition 61. Formulas of the language $\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$ are given by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi) \mid [a]\varphi \mid [!\sigma]\varphi,$$

with $p \in \mathsf{Prop}$, $i \in \mathsf{Agt}$, $a \in \mathsf{Act}$ and $\sigma \in \mathsf{Act}^*$. Formulas of the form $[!\sigma]\varphi$ are read as: *"after announcing that σ is distinguishable from any other plan, φ holds".*

Definition 62. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V}\rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$; take $\sigma \in \mathsf{Act}^*$ and $\varphi \in \mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$.

$$\mathcal{M}, w \models [!\sigma]\varphi \quad \textit{iff} \quad \mathcal{M}^\sigma, w \models \varphi,$$

where $\mathcal{M}^\sigma = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}', \mathrm{V}\rangle$, with

$$\mathrm{U}'(i) = \begin{cases} (\mathrm{U}(i) \setminus \{\pi\}) \cup \{\{\sigma\}\} & \text{if there is } \pi \in \mathrm{U}(i) \text{ such that } \sigma \in \pi, \\ \mathrm{U}(i) \cup \{\{\sigma\}\} & \text{otherwise.} \end{cases}$$

Intuitively, the new dynamic modality announces that a given plan has a unique characteristic. If the plan is not already under consideration by a certain agent, it is added to her set of plans, thereby making the agent aware of its existence. Otherwise, this modality replaces the set of plans containing σ with the singleton $\{\sigma\}$, eliminating any uncertainty around σ in the process. This stands in contrast to the refinement operators we considered before, which instead partition and preserve the original set of plans. Nevertheless, the agent's capacities are, in a sense, maintained, since σ remains to represent the set that originally contained it.

This new operator would be useful in the situation we discussed in Ex. 7.

Example 63. Recall the model $\mathcal{M}$ from Ex. 7. Now, suppose that a cooking professor teaches (announces) to both agents about the plan *ebmfsp*. Now every agent can distinguish the plan *ebmfsp* from all others. And, after the announcement, agent j knows how to make a good cake. Thus, $\mathcal{M} \models [!\mathit{ebmfsp}]\mathsf{Kh}_j(h, g)$.

The following proposition states that the new modality is self-dual and normal.

Proposition 64. *The following propositions hold:*

1. $\models [!\sigma]\varphi \leftrightarrow \neg[!\sigma]\neg\varphi$.
2. $\models [!\sigma](\varphi_1 \to \varphi_2) \to ([!\sigma]\varphi_1 \to [!\sigma]\varphi_2)$.
3. *If* $\models \varphi$, *then* $\models [!\sigma]\varphi$.

We proceed now to introduce a complete axiom system for $\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$. In this case, the expressive power of the underlying static logic is sufficient to capture the behavior of $[!\sigma]$, and we can obtain an axiomatization via reduction axioms.

Reduction axioms can be used to translate every formula of $\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$ into an equivalent formula of $\mathsf{L}_{\mathsf{Kh}_i,\Box}$. The challenging case is, as usual, to eliminate the occurrence of a $[!\sigma]$ with Kh_i under its scope. For simplicity, consider a formula $[!a]\mathsf{Kh}_i(\psi, \varphi)$ with $a \in \mathsf{Act}$. After a is announced to be different from any other plan, there are two possible reasons for $\mathsf{Kh}_i(\psi, \varphi)$ to be true. The first is that a plays no role in the truth of $\mathsf{Kh}_i(\psi, \varphi)$. In this case, we can push the dynamic modality into the front of the pre- and post-conditions of the Kh_i modality to obtain $\mathsf{Kh}_i([!a]\psi, [!a]\varphi)$. The second possibility is that distinguishing a creates new epistemic abilities. If the singleton $\{a\}$ is the witness for $\mathsf{Kh}_i(\psi, \varphi)$ it is because: 1) a is SE at every state satisfying ψ after the annoucement of a (notice that as a is a single action, SE is equivalent to executability); and 2) from every ψ-state, after every execution of a, we always get that φ holds (in the model updated by $[!a]$). The former is captured in the reduction axiom by $[!a]\psi \to \langle a\rangle\top$ holding everywhere, while the latter is captured by $[!a]\psi \to [a][!a]\varphi$. Putting all together, this case is reflected by $\mathsf{A}([!a]\psi \to (\langle a\rangle\top \wedge [a][!a]\varphi))$.

RAtom	$\vdash [!\sigma]p \leftrightarrow p$
R$\neg$	$\vdash [!\sigma]\neg\varphi_1 \leftrightarrow \neg[!\sigma]\varphi_1$
R$\vee$	$\vdash [!\sigma](\varphi_1 \vee \varphi_2) \leftrightarrow ([!\sigma]\varphi_1 \vee [!\sigma]\varphi_2)$
R$\Box$	$\vdash [!\sigma][a]\varphi_1 \leftrightarrow [a][!\sigma]\varphi_1$
RKh	$\vdash [!\sigma]\mathsf{Kh}_i(\varphi_1, \varphi_2) \leftrightarrow (\mathsf{Kh}_i([!\sigma]\varphi_1, [!\sigma]\varphi_2) \vee \mathsf{A}([!\sigma]\varphi_1 \to P(\sigma, [!\sigma]\varphi_2)))$
$\mathsf{RE}_{[!]}$	From $\vdash \varphi \leftrightarrow \psi$ derive $\vdash [!\sigma]\varphi \leftrightarrow [!\sigma]\psi$

Table 5: Reduction axioms $\mathcal{L}_{\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}}$.

This idea can be generalized when we consider an arbitrary plan σ instead of a single action. The case in which the announcement does not affect the knowledge is as before. For the other case, we need to make sure that if the announcement of σ guarantees that the initial condition holds ($[!\sigma]\varphi_1$), then σ is SE and after its execution, the goal is achieved ($[!\sigma]\varphi_2$). The latter conjunction is expressed by the predicate $P(\sigma, [!\sigma]\varphi_2)$ defined below. The complete list of reduction axioms for $\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$ are introduced in Table 5.

Proposition 65. *Let $\alpha \in \mathsf{Act}^*$, we define the extension $[\alpha]\psi$ of $[a]\psi$ as: $[\alpha]\psi = \psi$ if $\alpha = \epsilon$, and $[\alpha[1]] \ldots [\alpha[|\alpha|]]\psi$, otherwise. Thus, P is defined as:*

$$\begin{aligned} P(\epsilon, \psi) &= \psi \\ P(\alpha, \psi) &= (\textstyle\bigwedge_{k=0}^{|\alpha|-1}([\alpha_k]\langle\alpha[k+1]\rangle\top) \wedge [\alpha]\psi). \end{aligned}$$

Then, the reduction axioms from Table 5 are valid.

Proof. Let us focus only in the case of RKh. Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V}\rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$, then we prove below the two directions of the double implication.

($\Rightarrow$) Assume $\mathcal{M} \models [!\sigma]\mathsf{Kh}_i(\varphi_1, \varphi_2)$, i.e., $\mathcal{M}^\sigma \models \mathsf{Kh}_i(\varphi_1, \varphi_2)$ iff there is $\pi' \in \mathrm{U}'(i)$ s.t. $[\![\varphi_1]\!]^{\mathcal{M}^\sigma} \subseteq \mathrm{SE}^{\mathcal{M}^\sigma}(\pi')$ and $\mathrm{R}_{\pi'}([\![\varphi_1]\!]^{\mathcal{M}^\sigma}) \subseteq [\![\varphi_2]\!]^{\mathcal{M}^\sigma}$. Using $[\![\psi]\!]^{\mathcal{M}^\sigma} = [\![[!\sigma]\psi]\!]^{\mathcal{M}}$ and $\mathrm{SE}^{\mathcal{M}^\sigma}(\pi') = \mathrm{SE}^{\mathcal{M}}(\pi')$ (since the relations remain unchanged), this is equivalent to say that there is $\pi' \in \mathrm{U}'(i)$ s.t. $[\![[!\sigma]\varphi_1]\!]^{\mathcal{M}} \subseteq \mathrm{SE}^{\mathcal{M}}(\pi')$ and $\mathrm{R}_{\pi'}([\![[!\sigma]\varphi_1]\!]^{\mathcal{M}}) \subseteq [\![[!\sigma]\varphi_2]\!]^{\mathcal{M}}$. We need to consider two cases: If $\pi' \neq \{\sigma\}$, then $\pi \in \mathrm{U}(i)$ and we get $\mathcal{M} \models \mathsf{Kh}_i([!\sigma]\varphi_1, [!\sigma]\varphi_2)$. If $\pi' = \{\sigma\}$, then $[\![[!\sigma]\varphi_1]\!]^{\mathcal{M}} \subseteq \mathrm{SE}^{\mathcal{M}}(\sigma)$ and $\mathrm{R}_\sigma([\![[!\sigma]\varphi_1]\!]^{\mathcal{M}}) \subseteq [\![[!\sigma]\varphi_2]\!]^{\mathcal{M}}$. From the first part, using the definition of SE, we can derive that for all $w \in [\![[!\sigma]\varphi_1]\!]^{\mathcal{M}}$, $k \in [0 \ldots |\sigma| - 1]$ we have $\mathcal{M}, w \models [\sigma_l]\langle\sigma[l+1]\rangle\top$. From the second one, we can reach that for all $w \in [\![[!\sigma]\varphi_1]\!]^{\mathcal{M}}$, $\mathcal{M}, w \models [\sigma][!\sigma]\varphi_2$. Thus, for all $w \in \mathrm{W}$, $\mathcal{M}, w \models [!\sigma]\varphi_1 \to P(\sigma, [!\sigma]\varphi_2)$.

($\Leftarrow$) Conversely, suppose $\mathcal{M} \models \mathsf{Kh}_i([!\sigma]\varphi_1, [!\sigma]\varphi_2) \vee \mathsf{A}([!\sigma]\varphi_1 \to P(\sigma, [!\sigma]\varphi_2))$, then the disjunction is satisfied by either one of the following situations:

$\mathcal{M} \models \mathsf{Kh}_i([!\sigma]\varphi_1, [!\sigma]\varphi_2)$: Let $\pi \in \mathrm{U}(i)$ be s.t. $[\![[!\sigma]\varphi_1]\!]^{\mathcal{M}} \subseteq \mathrm{SE}^{\mathcal{M}}(\pi)$ and $\mathrm{R}_\pi([\![[!\sigma]\varphi_1]\!]^{\mathcal{M}}) \subseteq [\![[!\sigma]\varphi_2]\!]^{\mathcal{M}}$. As we already stated, this is equivalent to saying that $[\![\varphi_1]\!]^{\mathcal{M}^\sigma} \subseteq \mathrm{SE}^{\mathcal{M}^\sigma}(\pi)$ and $\mathrm{R}_\pi([\![\varphi_1]\!]^{\mathcal{M}^\sigma}) \subseteq [\![\varphi_2]\!]^{\mathcal{M}^\sigma}$. If $\sigma \notin \pi$, then $\pi \in \mathrm{U}'(i)$. If $\sigma \in \pi$, then $\{\sigma\} \in \mathrm{U}'(i)$, $\mathrm{SE}^{\mathcal{M}^\sigma}(\pi) \subseteq \mathrm{SE}^{\mathcal{M}^\sigma}(\{\sigma\})$ and $\mathrm{R}_{\{\sigma\}}([\![\varphi_1]\!]^{\mathcal{M}^\sigma}) \subseteq \mathrm{R}_\pi([\![\varphi_1]\!]^{\mathcal{M}^\sigma})$. In any case, we get $\mathcal{M} \models [!\sigma]\mathsf{Kh}_i(\varphi_1, \varphi_2)$.

$\mathcal{M} \models \mathsf{A}([!\sigma]\varphi_1 \to P(\sigma, [!\sigma]\varphi_2))$: Notice that σ meets the strong executability and reachability conditions. The only thing left to prove is that $\{\sigma\} \in \mathrm{U}'(i)$. But that is direct since whether $\sigma \in \pi$ for some $\pi \in \mathrm{U}(i)$ or not, in the end $\{\sigma\} \in \mathrm{U}'(i)$. Hence, $\mathcal{M} \models [!\sigma]\mathsf{Kh}_i(\varphi_1, \varphi_2)$. $\square$

Once more, from Prop. 65 and Thm. 53 we get:

Theorem 66. *The axiom system $\mathcal{L}_{\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}} + \mathcal{L}_{\mathsf{Kh}_i,\Box}$ for $\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$ is sound and strongly complete w.r.t. the class of all models.*

Finally, as the elimination of the dynamic modality takes a finite number of steps, using Cor. 60 we can conclude:

Corollary 67. *The satisfiability problem for $\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$ is decidable.*

It is easy to define a version of $[!\sigma]$ to handle semi-private announcements. $[!\sigma, i]\varphi$ is interpreted as *"after it is announced to agent i that the plan σ is distinguishable from any other plan, φ holds"*. Its semantics is given by

$$\mathcal{M}, w \models [!\sigma, i]\varphi \quad \textit{iff} \quad \mathcal{M}^{\sigma}_{i}, w \models \varphi,$$

where $\mathcal{M}^{\sigma}_{i} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}', \mathrm{V}\rangle$, with:

- $\mathrm{U}'(j) = \mathrm{U}(j)$ if $j \neq i$,
- $\mathrm{U}'(i) = (\mathrm{U}(i) \setminus \{\pi\}) \cup \{\{\sigma\}\}$, if there is $\pi \in \mathrm{U}(i)$ such that $\sigma \in \pi$,
- $\mathrm{U}'(i) = \mathrm{U}(i) \cup \{\{\sigma\}\}$ otherwise.

The resulting logic can be axiomatized via reduction axioms, where $[!\sigma, i]\mathsf{Kh}_i(\varphi_1, \varphi_2)$ can be eliminated via RKh_i in Table 5, while for $i \neq j$ we can simply add the axiom $\vdash [!\sigma, i]\mathsf{Kh}_j(\varphi_1, \varphi_2) \leftrightarrow \mathsf{Kh}_i([!\sigma, i]\varphi_1, [!\sigma, i]\varphi_2)$. With this at hand, we again obtain completeness and decidability.

6 Conclusions

In this article, we investigated dynamic modalities in the context of knowing how logics. Our starting point has been the uncertainty-based semantics from [30, 31], where indistinguishability is defined over plans instead of over states as in the framework of, e.g., [27, 29]. Building on this, we studied two kinds of model updates: ontic updates (via model operations for removing states and edges) and, more extensively, epistemic updates (via model operations for refining the agents' uncertainty over plans).

We show that, in general, the logics obtained when these operators are added to the basic knowing how logic $\mathsf{L}_{\mathsf{Kh}_i}$ have increased expressive power. As a result, the logics cannot be axiomatized by reduction axioms. In most cases, we were able to define a complete axiomatization using reduction axioms by restricting the class of models considered. We introduced the class of models $\mathbf{M_{BA}}$ (**BA** stands for 'basic actions') as the class of models $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathrm{U}, \mathrm{V}\rangle$ in which, for all $i \in \mathsf{Agt}$, we have that $\pi \in \mathrm{U}(i)$ implies $\pi \subseteq \mathsf{Act}$. $\mathbf{M_{BA}}$ could be interpreted as a more abstract representation of the abilities of the agents, where every plan is modeled as a single atomic action. This class bears resemblance to restricted classes considered in [49] for a local knowing how modality. Interestingly, while the basic knowing how logic $\mathsf{L}_{\mathsf{Kh}_i}$ cannot distinguish between the class of all $\mathrm{LTS}^{\mathrm{U}}$ and $\mathbf{M_{BA}}$, this is not the case in the presence of the dynamic operators we introduced.

For the ontic updates, we considered dynamic modalities well investigated in the DEL literature (see, e.g., [12, 35, 36]) that perform changes in the knowledge of the agents encoded by the LTS. These modalities take new meanings in the context of LTS^Us. Conceptually, these operations are natural choices, since part of the knowing how knowledge of an agent arises from the ability she has to access certain worlds through certain plans. These updates concern changes in the set of states of the model and in the accessibility relations. It is interesting to reflect on the impact that these updates have on the different logics discussed in the article. In DEL, updates on states and edges are clearly epistemic updates, as states other than the current state are situations the agent considers possible, and the edges linking them are indistinguishability relations connecting states the agent cannot discern. Hence, eliminating states and edges results in additional knowledge for the agents. In the original presentation of a knowing how operator with semantics defined over LTS, edges can be considered as encoding the agent's epistemic take on her abilities: the actions she considers possible, their preconditions and effects. Once more, removing states and edges has epistemic impact.

When LTSs are replaced by LTS^Us, states and edges encode ontic information. In other words, they stand for the actions available in the considered situation, their preconditions and effects, independently of the knowledge the agents have about them. Thus, in the uncertainty-based knowing how setting, the elimination of states and relations indicates not that they are no longer epistemically considered possible, but rather that they are no longer available in the current representation of the world. Thus, removing states produces an ontic change that might affect, indirectly, the agents' epistemic abilities.

Ultimately, the main focus of the article is on epistemic updates. We started by presenting an operation that distinguishes between two given plans and discussing some of its properties. In turn, this operator can be used to define one for arbitrary refinement. The issue of complete axiomatization (both over the class of all models and over some restricted class like $\mathbf{M_{BA}}$) is still elusive in this setting. Our final proposal for an epistemic dynamic modality is the operator $[!\sigma]$ that makes the plan σ distinguishable from any other, for all agents. In this case, we propose the language $\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$ that also extends the underlying static language with a normal modality for each basic action. For this logic, we provide sound and complete axiomatizations via reduction axioms over the class of all LTS^Us, and show that the satisfiability problem is decidable via filtrations.

This article should be considered as taking the first steps on a systematic study of dynamic operators in the setting of knowing how. It explores a number of proposals for both ontic and epistemic dynamic modalities. Among the introduced languages, $\mathsf{L}_{\mathsf{Kh}_i,\Box,[!\sigma]}$ is, to our knowledge, the first result of its class, showcasing an interesting proposal for a dynamic logic for knowing how, with an LTS^U-based semantics, with a sound and complete axiomatization and a decidable satisfiability problem.

Concerning the extension of the 'static' knowing how setting, one wonders the precise relationship, expressivity-wise, between the new language $\mathsf{L}_{\mathsf{Kh}_i,\Box}$ and other known languages (e.g., $\mathsf{L}_{\Box,\mathsf{A}}$). In fact, following that line of thought, one also wonders whether the Kh modality is definable from other 'more basic' operators (e.g., standard normal modalities plus nominals and/or operators quantifying over actions) and whether this would yield a more expressive language on which reduction axioms for the introduced dynamic modalities exist (see [42]). Also, we would like to characterize the exact complexity of the introduced dynamic logics.

A broader look is also possible, which would involve investigating other (maybe more general) dynamic operators that change the agents' epistemic abilities. For instance, one can use ideas from [50, 51] to define actions of semi-private ability-change.

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Declaration of conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Declaration of data availability

No new data were created or analyzed in this study.

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