Hybrid Logics

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Abstract

Hybrid languages are modal languages that have special symbols for naming individual states in models. Their history can be traced back to work of Arthur Prior in the fifties. The subject has recently regained interest, resulting in many new results and techniques. This chapter contains a modern overview of the field. We sketch its history, and survey the basic properties of various hybrid languages, focussing on model theory (completeness, expressivity, definability, interpolation), decidability and complexity, and proof theory. We also discuss a number of connections with other fields.

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This chapter provides a modern overview of the field of hybrid logic. Hybrid logics are extensions of standard modal logics, involving symbols that name individual states in models. The first results that are nowadays considered as part of the field date back to the early work of Arthur Prior in 1951. Since then, hybrid logic has gone through a number of revivals and reinventions. Nowaways, it is a field of research in its own right, with a wealth of results, techniques, and applications.

Our main aim, in this chapter, is to provide a coherent picture of the current state of affairs in the field of hybrid logic. Rather than a comprehensive summary, we will try to give the reader a taste for the type of results and techniques that we consider hallmarks of the field. In some cases, we will only state results, with pointers to relevant literature, while in other cases we will provide full proofs.

In Section 1, we give an intuitive introduction to hybrid logics, with examples of the extra expressive power offered by the hybrid operators. This section also contains the basic definitions of syntax and semantics that are used throughout the chapter. In Section 2 we provide a short history of the field, discussing the work of Prior in the 50s, of the Sofia School in the 80s, and the work on very expressive hybrid languages in the 90s. Sections 3 and 4 form the core of the chapter. They contain the most important techniques and results in the field, with respect to completeness, expressive power, frame definability, interpolation and complexity. In Section 5 we briefly present proof systems for hybrid languages (sequents, natural deduction, tableaux, and resolution), and we discuss some issues concerning the development of automated provers based on them. In Section 6 we comment on connections with related areas (some of which are discussed in detail in other chapters of this handbook). Section 7 finishes the chapter with a summary and general perspectives.

1 What are Hybrid Logics?

In their simplest form, hybrid languages are modal languages that have special symbols to name individual states in models. These new symbols, which are called *nominals*, enter the stage gracefully: we simply add a new sort of atomic symbols $\mathsf{NOM} = \{i, j, k, \ldots\}$ disjoint from the set PROP of propositional variables and let them combine freely in formulas. For example, if i is a nominal and p and q are propositional variables, then

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \to \Diamond(p \wedge q), \tag{1}$$

is a well formed formula. Now for the important twist: since nominals name individual states in the model, they denote *singleton sets*. In other words, they are true at a unique point in the model. Once this step has been taken, the whole landscape changes. For example, (1) becomes a validity: let $\mathcal{M}$ be a model, m a state in the domain of $\mathcal{M}$, and suppose $\mathcal{M}, m \models \Diamond(i \wedge p) \wedge \Diamond(i \wedge q)$. Then some successor state m' of m satisfies $i \wedge p$, and some successor state m'' of m satisfies $i \wedge q$. Since i is a nominal, it is true at a unique point in $\mathcal{M}$. Hence $m' = m''$ and we have $\mathcal{M}, m \models \Diamond(p \wedge q)$. Note that (1) could be falsified if i were an ordinary propositional variable.

When we realize the potential that nominals have, an interesting idea suggests itself: to introduce, for each nominal i, an operator $@_i$ that allows us to jump to the point named by i. The formula $@_i\varphi$ (read "at i, φ") moves the point of evaluation to the state named by i and evaluates φ there. These operators satisfy many nice logical properties. For a start, each $@_i$ is a normal modal operator: it satisfies the distributivity axiom $(@_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi))$ and the necessitation rule (if φ is valid, then $@_i\varphi$ is also valid). Moreover, it is self-dual: $@_i\varphi$ is equivalent to $\neg @_i \neg\varphi$. In an intuitive sense, the $@_i$ operators provide a bridge between semantics and syntax by internalizing the satisfaction relation '$\models$' into the logical language:

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M} \models @_i\varphi, \text{ where } i \text{ is a nominal naming } w.$$

For this reason, these operators are usually called *satisfaction operators*.

Aiming to make full use of the flexibility provided by direct reference to specific points in the model naturally leads to further enrichment of the language. One possibility would be to have not only names for individual states but also variables ranging over states, with corresponding quantifiers. We would then be able to write formulas like

$$\forall y.\Diamond y. \tag{2}$$

The first-order translation of this formula is $\forall y.\exists z.(R(x,z) \wedge z = y)$ or, simply, $\forall y.R(x,y)$, forcing the current state to be related to all states in the domain. The $\forall$ quantifier is very expressive. As discussed in [32], even the basic modal language extended with state variables and this universal quantifier is undecidable. Moreover, $\forall$ and @ together give us already full first-order expressive power (cf. Section 3.2). Nevertheless, the $\forall$ quantifier is historically important. The earliest treatments are probably those of [117,118,46].

The $\forall$ quantifier is very "classical." If we think modally, and remember that evaluation of modal formulas takes place *at a given point*, a different kind of binder suggests itself. The $\downarrow$ binder binds variables to points but, unlike $\forall$, it binds to the *current* point. In essence, it enables us to create a name for the here-and-now, and refer to it later in the formula. For example, the formula

$$\downarrow y.\Diamond y \tag{3}$$

is true at a state m iff m is related to itself. The intuitive reading of (3) is quite straightforward: the formula says "call the current state y and check that y is reachable." The difference between $\forall$ and $\downarrow$ is subtle, but important. $\forall$ is *global*, in the sense that formulas containing $\forall$ are not preserved under generated submodels [32]. On the other hand, $\downarrow$ is intrinsically local and, as we will show in Theorems 3.13 and 3.15, it can be characterized in terms of the operation of taking generated submodels.

Like $\forall$, the $\downarrow$ binder has been invented independently on several occasions. For example, in [122], $\downarrow$ is introduced as part of an investigation into temporal semantics and temporal databases, [131] uses it to aid reasoning about automata, and [52] employs it as part of his treatment of indexicality. However, none of the systems just mentioned allows the free syntactic interplay of variables with the underlying propositional logic; that is, they make use of $\downarrow$, but in languages that are not fully hybrid. The earliest paper to introduce it into a fully hybrid language seems to be [78].

Note that satisfaction operators work in perfect coordination with $\downarrow$. Whereas $\downarrow$ "stores" the current point of evaluation (by binding a variable to it), the satisfaction operators enable us to "retrieve" stored information by shifting the point of evaluation in the model. By using the "storing and retrieving" intuition it is easy to define complex properties. For example, Kamp's temporal *until* operator U (with semantics: $U(\varphi, \psi)$ is true at a state m if there is a future state m' where φ holds, such that ψ holds in all states between m and m') can be defined as follows:

$$U(\varphi, \psi) := \downarrow x.\Diamond\downarrow y.(\varphi \wedge @_x\Box(\Diamond y \to \psi)).$$

Let us see how this work. First, we name the current state x using $\downarrow$, and use the $\Diamond$ operator to find a suitable successor state, which we call y, where φ holds. Without the @ operator we would be stuck in that successor state, but we can use @ to go back to x and demand that in all successors of x having y as a successor, ψ holds.

Summarizing the above discussion, we can say that the term *hybrid logic* refers to a family of extensions of the basic hybrid language with devices that, in one way or another, allow for explicit reference to individual states of the Kripke model. But, why are hybrid logics called *hybrid*?

One explanation comes from the work of Arthur Prior in the 1950s. As we will discuss more in detail in Section 2, Prior was interested in the relation between what McTaggart called the A-series and B-series of time [109]. Following McTaggart's analysis of time in terms of the A-series of past, present and future and the B-series of earlier and later, Prior discusses two logical systems: the I-calculus aims to capture the properties of the B-series and takes variables ranging over instants as primitive, while the T-calculus examines tenses and takes variables ranging over propositions. In [117, Chapter V.6], Prior proposes a way to develop the I-calculus *inside* the T-calculus, and for this he allows instant-variables to be used together with propositional variables. He will call this step "the third grade of tense-logical involvement" in [118, Chapter XI], where instant-variables are treated as representing (special) propositions. From this perspective, the terms hybrid applies to the "confusion" of terms (the variables over instants) with formulas (the propositional variables).

There is another sense in which hybrid logics are hybrid, namely that, both in terms of expressive power and in terms of the techniques used to analyze them, hybrid languages lie in between the basic modal language and first-order logic. While having a distinctly modal flavor, hybrid logics enjoy features which are of a clear first-order nature. As we discussed above, the more expressive hybrid languages include binders and variables over elements of the domain, traditional hallmarks of first-order languages, while nominals are nothing else than first-order constants. The nominals and satisfaction operators also introduce a restricted form of equality: a state m in a model can satisfy a nominal i if and only if it is equal to the denotation of i, and a model $\mathcal{M}$ satisfies $@_i j$ if and only if the denotations of i and j coincide. In other words, nominals introduce equality between the point of evaluation and a named state, while satisfaction operators enable us to express equality between named states. Concerning first-order techniques which can be used for hybrid languages, we will see in Section 3.1 for example, that nominals can be used as 'witnesses' in a classical Henkin-style completeness proof for hybrid languages, and classical first-order notions like potential isomorphisms are useful for characterizing the expressive power of hybrid languages. And in Section 3.3, we will see a very general interpolation result, the proof of which relies on the fact that shared nominals can be "bound away" using $\downarrow$, in the same way that shared constants can be replaced by existentially quantified variables in first-order logic.

For a more detailed introduction, including further intuitive examples using the different hybrid languages, the reader is referred to [26]. The Hybrid Logic Web Pages [3] provides further information and a broad on-line bibliography. We now move on to the basic definitions of syntax and semantics that will be used through the chapter.

1.1 Basic Definitions

The simplest hybrid language is $\mathcal{H}$, which extends the basic modal language with nominals only. Further extensions will be named by listing the additional operators. The most expressive system we will discuss in detail is $\mathcal{H}(\mathsf{E}, @, \downarrow)$, with the existential modality E, @-operators, and the $\downarrow$ binder (when considering languages containing the $\downarrow$ binder, it is implicitly understood that the language also contains state variables). At various points, we will briefly mention other hybrid languages as well (e.g., hybrid extensions of temporal and dynamic logics).

The following two definitions give the syntax and semantics of $\mathcal{H}(\mathsf{E}, @, \downarrow)$. The corresponding definitions for sublanguages of $\mathcal{H}(\mathsf{E}, @, \downarrow)$ can be obtained by leaving out irrelevant clauses.

Definition 1.1 Let $\mathsf{REL} = \{R_1, R_2, \ldots\}$ (the *relational symbols*), $\mathsf{PROP} = \{p_1, p_2, \ldots\}$ (the *propositional variables*), $\mathsf{NOM} = \{i_1, i_2, \ldots\}$ (the *nominals*), and $\mathsf{SVAR} = \{x_1, x_2, \ldots\}$ (the *state variables*) be pairwise disjoint, countably infinite sets of symbols. By a *state symbol*, we will mean any element of $\mathsf{NOM} \cup \mathsf{SVAR}$. The well-formed formulas of the hybrid language $\mathcal{H}(\mathsf{E}, @, \downarrow)$ in the signature $\langle \mathsf{REL}, \mathsf{PROP}, \mathsf{NOM}, \mathsf{SVAR} \rangle$ are given by the following recursive definition:

$$\mathsf{FORMS} \ ::= \ \top \mid p \mid s \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle R \rangle \varphi \mid \mathsf{E}\varphi \mid @_s\varphi \mid \downarrow x.\varphi,$$

where $p \in \mathsf{PROP}$, $s \in \mathsf{NOM} \cup \mathsf{SVAR}$, $x \in \mathsf{SVAR}$, $R \in \mathsf{REL}$ and $\varphi, \varphi_1, \varphi_2 \in \mathsf{FORMS}$.

Given a set of formulas $\Gamma \subseteq \mathsf{FORMS}$, we will use $\mathsf{PROP}(\Gamma)$, $\mathsf{NOM}(\Gamma)$ and $\mathsf{SVAR}(\Gamma)$ to denote, respectively, the set of propositional variables, nominals, and state variables occurring in formulas in Γ. Also, for φ a formula, $\mathsf{SF}(\varphi)$ will be the set of subformulas of φ.

Note that the above syntax is simply that of ordinary (multi-modal) propositional modal logic extended with clauses for the state symbols and for $\mathsf{E}\varphi$, $@_s\varphi$ and $\downarrow x_j.\varphi$. Also, note that, like propositional variables, nominals and state variables can be used as atomic formulas. The difference between nominals and state variables is analogous to the difference between constants and variables in first-order logic: nominals cannot be bound by $\downarrow$, and their interpretation is specified by the model, whereas state variables are interpreted by assignment functions, and they can be bound by the $\downarrow$-binder.

The notions of *free* and *bound* state variable are defined as in first-order logic, with $\downarrow$ as the only binding operator. Similarly, other syntactic notions (such as *substitution*, and a state symbol t being *substitutable for x in φ*) are defined as in first-order logic. A *sentence* is a formula containing no free state variables. Furthermore, a formula is *pure* if it contains no propositional variables, and *nominal-free* if it contains no nominals.

In the remainder of the chapter we will assume fixed a signature $\langle \mathsf{REL}, \mathsf{PROP}, \mathsf{NOM}, \mathsf{SVAR}\rangle$. Now for the semantics.

Definition 1.2 A (hybrid) *model* $\mathcal{M}$ is a triple $\mathcal{M} = \langle M, (R^{\mathcal{M}})_{R\in\mathsf{REL}}, V\rangle$ such that M is a non-empty set, each $R^{\mathcal{M}}$ is a binary relation on M, and $V : \mathsf{PROP} \cup \mathsf{NOM} \to \wp(M)$ is such that for all nominals $i \in \mathsf{NOM}$, $V(i)$ is a singleton subset of M. We usually write M (roman letters) for the domain of a model $\mathcal{M}$, and call the elements of M *states*, *worlds* or *points*. Each $R^{\mathcal{M}}$ is an *accessibility relation*, and V is the *valuation*. A *frame* is defined in the usual way: as a model without a valuation. If $\mathcal{F} = \langle M, (R^{\mathcal{F}})_{R\in\mathsf{REL}}\rangle$ is a frame and V is a valuation on M, then $\mathcal{M} = \langle \mathcal{F}, V\rangle$ is the model $\langle M, (R^{\mathcal{F}})_{R\in\mathsf{REL}}, V\rangle$. In this case we, say that $\mathcal{M}$ *is based on* $\mathcal{F}$, and that $\mathcal{F}$ *is the underlying frame of* $\mathcal{M}$.

An *assignment* g for $\mathcal{M}$ is a mapping $g : \mathsf{SVAR} \to M$. Given an assignment $g : \mathsf{SVAR} \to M$, a state variable $x \in \mathsf{SVAR}$, and a state $m \in M$, we define g^x_m (an x*-variant of* g) by letting $g^x_m(x) = m$ and $g^x_m(y) = g(y)$ for all $y \neq x$.

Let $\mathcal{M} = \langle M, (R^{\mathcal{M}})_{R\in\mathsf{REL}}, V\rangle$ be a model, $m \in M$, and g an assignment for $\mathcal{M}$. For any state symbol $s \in \mathsf{NOM} \cup \mathsf{SVAR}$, let $[s]^{\mathcal{M},g}$ be the state denoted by s (i.e., for $i \in \mathsf{NOM}$, $[i]^{\mathcal{M},g}$ is the unique $m \in M$ such that $V(i) = \{m\}$, and for $x \in \mathsf{SVAR}$, $[x]^{\mathcal{M},g} = g(x)$). Then the *satisfaction relation* is defined as follows:

$$\begin{array}{lll}
\mathcal{M}, g, m \models \top & & \\
\mathcal{M}, g, m \models p & \text{iff} & m \in V(p) \quad \text{for } p \in \mathsf{PROP} \\
\mathcal{M}, g, m \models s & \text{iff} & m = [s]^{\mathcal{M},g} \quad \text{for } s \in \mathsf{NOM} \cup \mathsf{SVAR} \\
\mathcal{M}, g, m \models \neg\varphi & \text{iff} & \mathcal{M}, g, m \not\models \varphi \\
\mathcal{M}, g, m \models \varphi_1 \wedge \varphi_2 & \text{iff} & \mathcal{M}, g, m \models \varphi_1 \text{ and } \mathcal{M}, g, m \models \varphi_2 \\
\mathcal{M}, g, m \models \langle R\rangle\varphi & \text{iff} & \text{there is a state } m' \text{ such that } R^{\mathcal{M}}(m, m') \text{ and } \mathcal{M}, g, m' \models \varphi \\
\mathcal{M}, g, m \models \mathsf{E}\varphi & \text{iff} & \text{there is a state } m' \in M \text{ such that } \mathcal{M}, g, m' \models \varphi \\
\mathcal{M}, g, m \models @_s\varphi & \text{iff} & \mathcal{M}, g, [s]^{\mathcal{M},g} \models \varphi \quad \text{for } s \in \mathsf{NOM} \cup \mathsf{SVAR} \\
\mathcal{M}, g, m \models {\downarrow}x.\varphi & \text{iff} & \mathcal{M}, g^x_m, m \models \varphi.
\end{array}$$

The first six clauses in the definition of the satisfaction relation are similar to the ones for the basic modal language, except that they are relativized to an additional assignment function. Recall that nominals and state variables can be used as atomic formulas, in which case they act as propositional variables that are true at a unique state. The $\downarrow$ binder binds state variables to the state where evaluation is being performed (the *current world*), and $@_s$ shifts evaluation to the state named by s. As in first-order logic, if φ is a *sentence* (i.e., a formula with no free state variables), the truth of φ at a state in a model does not depend on the assignment. Hence, in this case we will write $\mathcal{M}, m \models \varphi$ instead of $\mathcal{M}, g, m \models \varphi$.

A formula φ is said to be *globally true* in a model $\mathcal{M}$ under an assignment g (notation: $\mathcal{M}, g \models \varphi$), if $\mathcal{M}, g, m \models \varphi$ for all $m \in M$. A formula φ is *satisfiable* if there is a model $\mathcal{M}$, an assignment g on $\mathcal{M}$, and a world $m \in M$ such that $\mathcal{M}, g, m \models \varphi$. A formula φ is *valid* (notation: $\models \varphi$) if for all models $\mathcal{M}$ and assignments g, $\mathcal{M}, g \models \varphi$. A formula φ is a *local consequence* of a set of formulas Σ if for all models $\mathcal{M}$, assignments g, and points $m \in M$, $\mathcal{M}, g, m \models \Sigma$ implies $\mathcal{M}, g, m \models \varphi$. A formula φ is a *global consequence* of a set of formulas Σ if for all models $\mathcal{M}$ and assignments g, $\mathcal{M}, g \models \Sigma$ implies $\mathcal{M}, g \models \varphi$. We denote local consequence by $\Sigma \models^{loc} \varphi$ and global consequence by $\Sigma \models^{glo} \varphi$. As in ordinary propositional modal logic, local consequence is strictly stronger than global consequence.

Definitions 1.1 and 1.2 specify the syntax and semantics of the most expressive hybrid language we are going to discuss in detail, $\mathcal{H}(\mathsf{E}, @, \downarrow)$. Two important fragments of this language are $\mathcal{H}(@, \downarrow)$, which is obtained by dropping the clauses for the existential modality E, and $\mathcal{H}(@)$, which is obtained by dropping in addition the state variables and the $\downarrow$-binder. In other words, $\mathcal{H}(@)$ is simply the extension of the basic modal language with nominals and satisfaction operators. The languages $\mathcal{H}(@)$ and $\mathcal{H}(@, \downarrow)$ will receive most attention in this chapter. In sublanguages of $\mathcal{H}(\mathsf{E}, @, \downarrow)$ not containing $\downarrow$, variables and assignments play no role and are dropped from the above definitions.

2 History

In this section we will provide an overview of the historical development of hybrid languages, starting with the pioneering work of Prior, through the "revival" in the late eighties and early nineties in Sofia, and ending with the work of Blackburn and Seligman in the late nineties.

2.1 The Foundational Work of Prior

The work of Prior in modal logic and in particular in the modal analysis of time is well known, to the point that he is usually regarded as the inventor of temporal logic. For a detailed discussion of Prior's contributions to this field, together with some biographical information, see [111]. The following discussion is based on [51], a short but very good overview. See also [27], especially Section 4.

Prior is considered one of the most important promoters of the application of modal syntax to the formalisation of a wide variety of phenomena. Less well known is the fact that Prior, in collaboration with Carew Meredith, devised a version of possible worlds semantics roughly at the same time as, but independently of, the work of Carnap on modal semantics and several years before Kripke published his first paper on the topic. Interestingly, this part of Prior's work is already closely related to hybrid logic.

Nowadays, the view that modal logic can be seen as a fragment of first-order or second-order logic is commonplace. This is fairly straightforwards once we observe the possible worlds semantics of modal operators. When reading the earlier work of Prior, however, we should keep in mind that, at that time, most modal intuitions came solely from axiomatics. Nevertheless, in Prior's (unpublished) second book "The Craft of Formal Logic" (completed in 1951) we can find the following passage:

> For the similarity in behaviour between signs of modality and signs of quantity, various explanations may be offered. It may be, for example, that signs of modality are just ordinary quantifiers operating upon a peculiar subject-matter, namely possible states of affairs... It would not be quite accurate to describe theories of this sort as "reducing modality to quantity." They do reduce modal *distinctions* to distinctions of quantity, but the variables to which the quantifiers are attached retain something modal in their signification — they signify "possibilities", "chances", "possible states of affairs", "possible combinations of truth-values", or the like.

Two things should be noticed in this passage. Firstly, the reference to "possible states of affairs" and even "possible combinations of truth-values," is a very early reference to possible worlds semantics. Secondly, Prior's strong reservations concerning "reducing modality to quantity". This early intuition on the foundational nature of modality later grew into a mature philosophy in Prior's view that quantification over possible worlds and instants was to be interpreted in terms of modality and tense — which constituted primitive notions — and not vice versa (although he did recognized that the study of both quantity and modality could benefit of each other).

Three years later, in 1954 at the New Zealand Congress of Philosophy, Prior presented a paper (not published until much later as [116]) in which his philosophical position is made more explicit. Working already in the framework of temporal logic, he introduces in this paper the I-calculus (which he will later call the U-calculus). In the I-calculus, propositions of the tense calculus are treated as predicates expressing properties of dates (which are represented by variables). The formula px should be read as "p at x," and I is a binary relation taking dates as arguments where Ixy is read as "y is later than x." Using an arbitrary date x to represent the time of utterance, Fp (intuitively, "the proposition p happens in the future") is equated with $\exists y.(Ixy \wedge py)$ (i.e., "p at some time later than x") and similarly for Pp, "the proposition p happens in the past." Prior mentions already that, by imposing various conditions on the relation I, analogues of the axioms of the tense calculus can be derived in the I-calculus.

Later in the same paper, Prior includes a detailed warning against regarding this interpretation of the tense calculus within the I-calculus as "a metaphysical explanation of what we mean by *is*, *has been* and *will be*"; he stresses that the I-calculus is not "metaphysically fundamental." He explains that F(Socrates is sitting down) means "It is *now* the case that it will be the case that Socrates is sitting down," and there is no genuine way of representing the indexical *now* in the I-calculus (he says that the free variable x is "a complete sham"). He continues: "If there is to be any 'interpretation' of our calculi in the metaphysical sense, it will probably need to be the other way round; that is, the I-calculus should be exhibited as a logical construction out of the PF-calculus rather than *vice versa*." This idea of the primacy of the tense calculus over the I-calculus — or, as he was later to put it, of McTaggart's A-series over the B-series, see [109] — was to become a central and distinctive tenet of his philosophy. These issues form the theme of his final, unfinished, book [119], but they already appear in some earlier articles.

But of course, the reconstruction of the I-calculus within the tense calculus is impossible, as the I-calculus is strictly more expressive than the tense calculus. Prior recognized this fact and investigated ways to extend the expressive power of the tense calculus to permit the reconstruction. This directly led to what we call today *very expressive hybrid languages* (i.e., hybrid languages including the $\forall$ binder). In [117, Chapter V.6], he actually proposes a way to develop the I-calculus inside the tense calculus, and for this he allows instant variables to be used together with propositional variables. He will call this step "the third grade of tense-logical involvement" in [118, Chapter XI], where instant variables are treated as representing (special) propositions.

We see, then, that Prior's development of hybrid languages was rooted in his philosophical convictions, and was instrumental in the implementation of some of his very early intuitions on time and tense. Prior's death in 1969 put an end to these investigations. Notice though, that Prior was never fully satisfied with his solution. It was technically correct (and actually quite bold and ingenious) but he was concerned that, in managing to "upgrade" the tense calculus to full first-order expressivity, the language had lost its claim to a metaphysical fundamentality. Robert Bull, a student of Prior, pushed the ideas of hybridization further in [46], where he provides an axiomatization and completeness result for a logic containing variables for *paths* on a model, which he calls "history-propositional" variables.

2.2 The Sofia School

As we saw, the roots of hybrid logic go back to Prior and Bull. About fifteen years later in Sofia, Bulgaria, nominals were re-discovered by Gargov, Passy and Tinchev in their investigations on Boolean modal logic and propositional dynamic logic. One of the issues that led them into these investigations was the following asymmetry in the expressive power of the modal language. The union of two accessibility relations is definable in the basic modal language, in the sense that the formula

$$\langle T\rangle p \leftrightarrow \langle R\rangle p \vee \langle S\rangle p$$

is valid on a frame precisely if the accessibility relation interpreting $\langle T\rangle$ is the union of the accessibility relations interpreting $\langle R\rangle$ and $\langle S\rangle$. Moreover, when added to the basic modal language, this formula completely axiomatizes the modal logic of the relevant class of frames.

Surprisingly, *intersection* of accessibility relations is not definable in the same way: it follows from the Goldblatt-Thomason theorem [77] that there is no formula in the basic modal language that is valid on a frame precisely if the accessibility relation of $\langle T\rangle$ is the intersection of the accessibility relation of $\langle R\rangle$ and $\langle S\rangle$. And even though the axiom scheme $\langle T\rangle p \rightarrow \langle R\rangle p \wedge \langle S\rangle p$ (together with the standard axioms and rules for the basic polymodal logic) completely axiomatizes the logic of this frame class, it is valid on the larger class where the accessibility relation of $\langle T\rangle$ is contained in the intersection of the accessibility relation of $\langle R\rangle$ and $\langle S\rangle$.

Now, Gargov, Passy and Tinchev showed in [75] that intersection *can* be defined using *nominals*. Indeed, for i a nominal, the axiom scheme

$$\langle T\rangle i \leftrightarrow \langle R\rangle i \wedge \langle S\rangle i$$

defines intersection in the above sense, and exactly axiomatizes the logic of the relevant class of frames (when added to an appropriate base axiomatization) [1]. The same story goes for complementation: there is no formula

[1] Note that this implies that the Goldblatt-Thomason theorem, in its usual form, does not hold for hybrid languages.

of the basic modal language that is valid on a frame precisely if the accessibility relation of $\langle R\rangle$ is the complement of the accessibility relation of S, but such a formula exists when nominals are added to the language: $\langle R\rangle i \leftrightarrow \neg\langle S\rangle i$.

This form of capturing the Boolean operations (together with an alternative based on the "sufficiency operator" $\boxplus$) was investigated by Gargov, Passy and Tinchev in [75]. In that paper, the first complete axiomatization of the minimal hybrid language is given. Following [76], recursively define $\Box$- and $\Diamond$-forms as follows: 1) \$ is both a $\Box$- and a $\Diamond$-form (where \$ is a fixed symbol not occurring in the language); 2) If L is a $\Box$-form and φ a formula, then $(\varphi \to L)$ and $\Box L$ are also $\Box$-forms; and 3) If M is a $\Diamond$-form and φ is a formula, then $(\varphi \wedge M)$ and $(\Diamond M)$ are also $\Diamond$-forms. For F a $\Box$- or $\Diamond$-form and φ a formula, let $F(\varphi)$ be the formula obtained by replacing the unique occurrence of \$ in F by φ. Now, Gargov, Passy and Tinchev showed that any complete axiomatization of the basic modal language, extended with the axioms

$$M(i \wedge \varphi) \to L(i \to \varphi) \quad \text{for } i \text{ a nominal, } L \text{ a } \Box\text{-form and } M \text{ a } \Diamond\text{-form}$$

completely axiomatizes the hybrid logic (in the language $\mathcal{H}$) of the class of all frames.

Besides the minimal hybrid language $\mathcal{H}$, Gargov, Passy and Tinchev also studied a richer hybrid language, obtained by extending propositional dynamic logic (PDL, cf. Chapter **??** of this handbook) with nominals. Intersection of accessibility relations is particularly interesting in this setting, as it can be interpreted as parallelism, or concurrency of programs. Passy and Tinchev [113] propose an extension of PDL with nominals and the universal modality, which they call Combinatory PDL (CPDL). The paper contains an axiomatization of CPDL($\cap, ^-, \subset, ^{-1}$), combinatory PDL extended with program intersection, complementation, subprograms and inverse, shown in Figure 1. Note that this axiomatization contains an infinitary rule (R2), i.e., an inference rule with infinitely many premises.

Besides the standard axioms and rules of PDL, and the axioms for the universal program ν, notice the definitions of union (A8), intersection (A9), complement (A10), subprogram (A11) and inverse program (A12). Notice also how the presence of the universal program ν helps defining the behaviour of nominals in axioms (A1) and (A2). Finally, notice the "Gabbay-Burgess-style rule" (R1) [68], which ensures that models are named, i.e., each state in the model is the denotation of some nominal (this also implies that the models are countable). Axiomatizations for sublanguages of CPDL($\cap, ^-, \subset, ^{-1}$) are obtained by dropping the corresponding definitions of the absent operators. In particular CPDL, "core" combinatory PDL, is axiomatized by axioms (A1) to (A8), (A13) to (A15) and rules (R1) to (R4) [2].

Passy and Tinchev proved a number of interesting properties of CPDL (see [115] for further details). For example, they observed that named models (i.e., models in which each state is named by a nominal) can be completely described by a set of formulas of the form $(\neg)@_i p$, $(\neg)@_i\Diamond j$ or $(\neg)@_i j$. Clearly, this property only depends on the expressive power of nominals and @, and hence holds already for $\mathcal{H}(@)$. This observation provides the theoretical basis for automated theorem proving and model building via the definition of Herbrand models (i.e., a model can be represented by the set of elementary formulas which are true in it, see [17]).

With respect to (un)decidability results, naturally the negative results concerning the undecidability of both global and local consequence in PDL [86] transfers to CPDL. Passy and Tinchev provide some (un)decidability results for satisfiability of languages related to CPDL in [115], while Gargov provides in [72] a finitary axiomatization of CPDL and proves the finite model property and decidability of the satisfiability problem for CPDL. Actually, the complexity of satisfiability in CPDL coincides with the one in PDL, EXPTIME-complete [56,55].

Theorem 2.1 *For $\Gamma \cup \{\varphi\}$ a decidable set of CPDL formulas, deciding whether $\Gamma \models^{glo} \varphi$ and $\Gamma \models^{loc} \varphi$ is Π^1_1-complete. On the other hand, satisfiability of CPDL formulas is* EXPTIME*-complete.*

Gargov's axiomatizability result mentioned above uses Segerberg's axiom $\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$ to replace the (R2) rule and shows that the (R1) rule is redundant, but infinitary rules cannot always be eliminated. For example, satisfiability of CPDL($^-$) is highly undecidable (Σ^1_1-complete) from which it follows that no

[2] Actually, in [115], the infinitary version of (R1) "If $\vdash [\alpha]\neg i$ for all $i \in \mathsf{NOM}$ then $\vdash [\alpha]\bot$" is discussed, which is necessary for completeness in some extensions of CPDL.

Axiom Schemes:

(A0)	All propositional tautologies
(A1)	$\langle \nu \rangle i$
(A2)	$\langle \nu \rangle (i \wedge \varphi) \rightarrow [\nu](i \rightarrow \varphi)$
(A3)	$\varphi \rightarrow \langle \nu \rangle \varphi$
(A4)	$\langle \nu \rangle \langle \nu \rangle \varphi \rightarrow \langle \nu \rangle \varphi$
(A5)	$\varphi \rightarrow [\nu] \langle \nu \rangle \varphi$
(A6)	$\langle \alpha \rangle \varphi \rightarrow \langle \nu \rangle \varphi$
(A7)	$\langle \alpha \beta \rangle \varphi \rightarrow \langle \alpha \rangle \langle \beta \rangle \varphi$
(A8)	$\langle \alpha \cup \beta \rangle i \leftrightarrow \langle \alpha \rangle i \vee \langle \beta \rangle i$
(A9)	$\langle \alpha \cap \beta \rangle i \leftrightarrow \langle \alpha \rangle i \wedge \langle \beta \rangle i$
(A10)	$\langle \bar{\alpha} \rangle i \leftrightarrow [\alpha] \neg i$
(A11)	$\alpha \subset \beta \leftrightarrow [\alpha \cap \bar{\beta}] \bot$
(A12)	$\langle \nu \rangle (i \wedge \langle \alpha^{-1} j \rangle) \leftrightarrow \langle \nu \rangle (j \wedge \langle \alpha \rangle i)$
(A13)	$\langle \varphi ? \rangle \psi \leftrightarrow \varphi \wedge \psi$
(A14)	$\langle \alpha^* \rangle \varphi \leftrightarrow \varphi \vee \langle \alpha \rangle \langle \alpha^* \rangle \varphi$
(A15)	$[\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$

Rules:

(R1)	If $\vdash [\alpha]\neg i$ for some i not in α, then $\vdash [\alpha]\bot$.
(R2)	If $\vdash [\beta][\alpha^i]\varphi$ for all $i \in \mathbb{N}$, then $\vdash [\beta][\alpha^*]\varphi$.
(R3)	If $\vdash \varphi$, then $\vdash [\nu]\varphi$.
(R4)	If $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

Where φ, ψ are formulas, α, β programs, ν the universal program and i, j nominals.

Fig. 1. Axiomatization of CPDL($\cap, ^-, \subset, ^{-1}$)

finitary axiomatization can be complete. Passy and Tinchev [115] discuss the issue of eliminability of the infinitary rules in detail (cf. also [98] for more recent results on infinitary axiomatizations of hybrid logics).

We now move into more expressive hybrid languages similar to those used by Prior and Bull. Chapter III of [115] is devoted to CDL, Combinatory Dynamic Logic which allows quantification over state variables. Interestingly, the authors seem to present CDL as an alternative to quantified modal logic, stating that replacing classical quantification (over the domains in each state of the model) by hybrid quantification (over the states themselves) leads to a better behaved system. While this is true, it also leads to a system which does not resemble quantified modal logic! In any case, it is interesting to see that, once nominals have been discovered, explicit quantification over states becomes a natural extension.

The following complete axiomatization of CDL is given in [115]:

	All axioms and rules of CPDL minus (R1), plus
(A16)	$\exists c.c$
(A17)	$\forall c.\varphi \rightarrow \varphi[c/d]$
(A18)	$\forall c.[\alpha]\varphi \rightarrow [\alpha]\forall c.\varphi$ for c with no free occurrences in α.
(R5)	If $\vdash \varphi$, then $\vdash \forall c.\varphi$.

The Sofia tradition in hybrid logics continues with the work of Goranko. In [73], Gargov and Goranko investigate the basic modal language extended first with nominals and the universal and existential modalities ($\mathcal{H}(\mathsf{E})$), and then with the difference operator D ($\mathcal{ML}(\mathsf{D})$)[3]. They prove that both languages are equivalent

[3] The semantic condition for the difference operator D is $\mathcal{M}, w \models \mathsf{D}\varphi$ iff there is a $w' \neq w$ such that $\mathcal{M}, w' \models \varphi$.

with respect to frame definability, and then provide characterizations of frame definability for these languages.

The work of Gargov and Goranko is historically relevant because, within the Sofia school, it marks the start of research on hybrid logics as such, and not as part of their research on extensions of PDL. Around the same time, but independently, Blackburn was studying simple hybrid languages over a Prior-style tense logic [21,22]. These two lines of research can be considered the origins of the current perspective on hybrid logics.

Goranko is also the first to investigate the $\downarrow$ binder in the context of hybrid logic. In [78], he extends the basic modal language with the universal modality and the $\downarrow$ binder with only a single state variable (though using a slightly different notation). Goranko provides an axiomatization for this logic, and illustrations of its high expressivity (sufficient, for example, to define Kamp's $U(p,q)$ and $S(p,q)$ and Stavi's $U'(p,q)$ and $S'(p,q)$ temporal operators and to simulate Prior's instant variables), and shows that the satisfiability problem for this language is undecidable. He mentions in the same paper that introducing multiple state variables would be possible, and investigates the resulting language in more detail in [79].

In [80], Goranko uses hybrid binders to design CTL_{rp} (CTL with reference pointers), a computation tree logic for finitely branching ω^+-trees, and defines syntactic and semantic interpretations between CTL^* and CTL_{rp}. In particular, this yields a complete axiomatization for the translations of all valid CTL^*-formulas, a step forwards in the search for a complete direct axiomatization of CTL^*, a long standing open problem finally solved in [121].

With this we conclude our (necessarily brief) overview of the work on hybrid logics done by the Sofia School. It is interesting to note that most of the languages studied by the Sofia school included the universal modality. In the following years and mainly through the work of Blackburn and Seligman, research in hybrid languages deals with, on the one hand, weak languages containing only nominals (e.g., [23,33]) and, on the other hand, very expressive languages containing binders (e.g., [31,35,37]).

2.3 Very Expressive Hybrid Languages

In the mid-nineties, Blackburn and Seligman [31] studied a number of very expressive hybrid languages, obtained by means of various state variable binders. We will review a few of these binders here, most of which will not return in the remainder of the chapter.

Up to now, we have introduced two hybrid binders, the "classical" $\exists$ and the "more modal" $\downarrow$. Let us review their semantic definitions. Given a model $\mathcal{M} = \langle M, (R^{\mathcal{M}})_{R \in \mathsf{REL}}, V\rangle$, an assignment g in $\mathcal{M}$ and $m \in M$:

$$\begin{aligned} \mathcal{M}, g, m \models \exists x.\varphi \;&\text{ iff }\; \mathcal{M}, g^x_{m'}, m \models \varphi \text{ for some } m' \in M. \\ \mathcal{M}, g, m \models {\downarrow} x.\varphi \;&\text{ iff }\; \mathcal{M}, g^x_m, m \models \varphi. \end{aligned}$$

Both quantifiers let us change the value assigned to x, without changing the point of evaluation. In [31] Blackburn and Seligman investigate two other binders which, besides changing the value of the bound variable, also change the point of evaluation:

$$\begin{aligned} \mathcal{M}, g, m \models \Sigma x.\varphi \;&\text{ iff }\; \mathcal{M}, g^x_{m'}, m' \models \varphi \text{ for some } m' \in M. \\ \mathcal{M}, g, m \models {\Downarrow} x.\varphi \;&\text{ iff }\; \mathcal{M}, g^x_m, m' \models \varphi \text{ for some } m' \in M. \end{aligned}$$

It is not hard to see that $\Sigma x.\varphi$ is equivalent to $\mathsf{E}{\downarrow} x.\varphi$, whereas ${\Downarrow} x.\varphi$ is equivalent to ${\downarrow} x.\mathsf{E}\varphi$. The Standard Translation (cf. Chapter **??** of this handbook) may be extended to these hybrid languages, in which case the appropriate clauses for these operators would be as follows (we provide also the clause for E for comparison):

$$\begin{aligned}
ST_x(\mathsf{E}\varphi) &= \exists z.ST_z(\varphi) \quad (z \text{ a variable not in } \varphi)\\
ST_x(\exists y.\varphi) &= \exists y.ST_x(\varphi)\\
ST_x(\downarrow y.\varphi) &= \exists y.(y = x \wedge ST_x(\varphi))\\
ST_x(\Sigma y.\varphi) &= \exists y.ST_y(\varphi)\\
ST_x(\Downarrow y.\varphi) &= \exists z.\exists y.(y = x \wedge ST_z(\varphi)) \quad (z \text{ a variable not in } \varphi).
\end{aligned}$$

The main result in [31] is that these binders form an expressive hierarchy. If we let $<$ stand for the relation "is strictly less expressive than" then we have that $\mathcal{H}(\downarrow) < \mathcal{H}(\exists) < \mathcal{H}(\Downarrow)$ and $\mathcal{H}(\mathsf{E}) < \mathcal{H}(\Sigma) < \mathcal{H}(\Downarrow)$. The expressivity inclusions are proved using the following equivalences:

$$\begin{aligned}
\downarrow x.\varphi &\equiv \exists x.(x \wedge \varphi)\\
\exists x.\varphi &\equiv \Downarrow z.\Downarrow x.(z \wedge \varphi) \quad (z \text{ a variable not in } \varphi)\\
\mathsf{E}\varphi &\equiv \Sigma z.\varphi \quad (z \text{ a variable not in } \varphi)\\
\Sigma x.\varphi &\equiv \Downarrow z.\Downarrow x.(x \wedge \varphi) \quad (z \text{ a variable not in } \varphi).
\end{aligned}$$

Moreover, the equivalence $\Downarrow x.\varphi \equiv \downarrow x.\mathsf{E}\varphi$ shows that $\mathcal{H}(\Downarrow) \leq \mathcal{H}(\downarrow, \mathsf{E})$ and hence any language containing an operator from each of the two "branches" in the hierarchy is expressively equivalent to $\Downarrow$. The strictness of the hierarchy is proved in [31] using different variants of bisimulations, preserving truth of formulas of the various languages.

In [141,142], Tzakova explores some examples of very expressive hybrid languages with binding operators in more detail, both axiomatically and by means of tableaux systems.

We turn now from motivation and historical remarks to recent developments and the current state of the field.

3 Model Theory

Many different hybrid languages were introduced in the previous sections. In this section, we will discuss two languages in more detail, namely $\mathcal{H}(@)$ and $\mathcal{H}(@, \downarrow)$. These two hybrid languages have received most attention in recent literature, and the proofs of the results we will discuss can usually be adapted to other hybrid languages.

3.1 Completeness

One of the most important motivations for the study of hybrid logics has been that the addition of nominals to the modal language makes it possible to prove very general completeness results, using a straightforward adaptation of the Henkin construction for first-order logic.

Definition 3.1 The logic $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$ is the smallest set of $\mathcal{H}(@, \downarrow)$ formulas that includes all axioms, and is closed under the rules, given in Figure 2. Given a set Σ of $\mathcal{H}(@, \downarrow)$ formulas, $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$ is the logic obtained by adding all formulas in Σ as axioms to $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$, and closing again under the rules in Figure 2. Given a set of $\mathcal{H}(@)$-formulas Σ, $\mathbf{K}_{\mathcal{H}(@)}$ and $\mathbf{K}_{\mathcal{H}(@)} + \Sigma$ are defined analogous to $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$ and $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$, except without the DA axiom scheme (note that this is the only axiom or rule in which $\downarrow$ occurs).

One note should be made, concerning the substitution rule (Subst). By this rule, one cannot only replace propositional variables uniformly by arbitrary formulas, but one can also replace nominals uniformly by other nominals (note that substituting nominals by formulas does not preserve validity in general).

Axioms:

(CT)	All classical tautologies
($K_\Box$)	$\vdash [R](\varphi \to \psi) \to [R]\varphi \to [R]\psi$
($K_@$)	$\vdash @_i(\varphi \to \psi) \to @_i\varphi \to @_i\psi$
($\text{Selfdual}_@$)	$\vdash @_i\varphi \leftrightarrow \neg @_i \neg \varphi$
($\text{Ref}_@$)	$\vdash @_i i$
(Agree)	$\vdash @_i @_j \varphi \leftrightarrow @_j \varphi$
(Intro)	$\vdash i \to (\varphi \leftrightarrow @_i\varphi)$
(Back)	$\vdash \langle R \rangle @_i\varphi \to @_i\varphi$
(DA)	$\vdash @_i(\downarrow x.\varphi \leftrightarrow \varphi[x/i])$

Rules:

(MP)	If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$
(Subst)	If $\vdash \varphi$ then $\vdash \varphi^\sigma$, for σ a substitution
($\text{Gen}_@$)	If $\vdash \varphi$ then $\vdash @_i\varphi$
($\text{Gen}_\Box$)	If $\vdash \varphi$ then $\vdash [R]\varphi$
(Name)	If $\vdash @_i\varphi$ and i does not occur in φ, then $\vdash \varphi$
(BG)	If $\vdash @_i\langle R \rangle j \to @_j\varphi$, $j \neq i$ and j does not occur in φ, then $\vdash @_i[R]\varphi$

Fig. 2. Axioms and rules for $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$

We call an axiomatization *complete* with respect to a class of frames, if for all formulas φ of the relevant language, φ is derivable in the axiomatization iff φ is valid on the given frame class. An axiomatization is *strongly complete* with respect to a frame class if for every set of formulas Σ and formula φ of the relevant language, $\Sigma \models^{loc} \varphi$ iff there are $\psi_1, \ldots, \psi_n \in \Sigma$ such that $\psi_1 \wedge \cdots \wedge \psi_n \to \varphi$ is derivable.

The following completeness result is taken from [34], but slight variations of it can be found already in [37]. Recall that a formula is *pure* if it contains no propositional variables (but may possibly contain nominals).

Theorem 3.2 (Pure completeness)

(i) *Let Σ be any set of pure $\mathcal{H}(@)$-formulas. Then $\mathbf{K}_{\mathcal{H}(@)} + \Sigma$ is strongly complete for the class of frames defined by Σ.*

(ii) *Let Σ be any set of pure $\mathcal{H}(@,\downarrow)$-formulas. Then $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$ is strongly complete for the class of frames defined by Σ.*

By *the frame class defined by* Σ, we mean the class of frames on which each formula in Σ is valid. Many frame properties can be defined using pure hybrid formulas, including properties such as irreflexivity, that cannot be defined in the basic modal language. A precise characterization of frame properties definable by pure formulas will be given in Section 3.2.3.

The proof of Theorem 3.2 trades heavily on the presence of the (Name) and (BG) rules. In [34], Blackburn and ten Cate show that, in the case of $\mathcal{H}(@,\downarrow)$, these rules (which are non-orthodox in the sense that they involve syntactic side conditions) can be replaced by

($\text{Name}_\downarrow$)	$\vdash \downarrow s.(s \to \varphi) \to \varphi$	provided that s does not occur in φ
($\text{BG}_\downarrow$)	$\vdash @_i[R]\downarrow s.@_i\langle R \rangle s$	
($\text{Gen}_\downarrow$)	If $\vdash \varphi$ then $\vdash \downarrow s.\varphi$.	

and an axiomatization with only orthodox rules is obtained, for which Theorem 3.2 still holds. In the case of $\mathcal{H}(@)$, on the other hand, the (Name) and (BG) rule cannot be eliminated. More precisely, every axiomatization for $\mathcal{H}(@)$ that is complete for arbitrary pure extensions contains either infinitely many rules or rules with side conditions [34].

Part of the present section will be devoted to a proof of Theorem 3.2. However, before we start, we will mention some other, complementary completeness results.

Theorem 3.2 resembles in spirit the Sahlqvist completeness theorem for modal logic (cf. Chapter **??** of this handbook). This raises the question of how pure formulas and Sahlqvist formulas relate, both in terms of expressive power and in terms of proof theoretic behaviour. As it turns out, for every modal Sahlqvist formula φ there is a pure sentence ψ of $\mathcal{H}(@, \downarrow)$ that defines the same frame class as φ, and, moreover, ψ can be picked such that $\varphi \to \psi$ is provable in $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$ [4]. It follows from this observation that every extension of $\mathbf{K}_{\mathcal{H}(@,\downarrow)}$ with modal Sahlqvist axioms is complete.

However, there are frame properties that can be defined by modal Sahlqvist formulas but not by pure $\mathcal{H}(@)$-formulas. For example, no set of pure $\mathcal{H}(@)$-formulas defines the same frame class as the modal Sahlqvist formula (CR) $\Diamond\Box p \to \Box\Diamond p$. This makes the following result, proved in [140], interesting.

Theorem 3.3 (Sahlqvist completeness) *Let Σ be any set of modal Sahlqvist formulas. Then $\mathbf{K}_{\mathcal{H}(@)} + \Sigma$ is strongly complete for the class of frames defined by Σ.*

Completeness does not hold for arbitrary combinations of pure formulas and modal Sahlqvist formulas. Consider the Sahlqvist axiom (CR) given above and the pure formula (NoGrid) $\Diamond(i \wedge \Diamond j) \to \Box(\Diamond j \to i)$. The incompleteness of $\mathbf{K}_{\mathcal{H}(@)} + \{\text{(CR)}, \text{(NoGrid)}\}$ is proved in [140] using a general frame argument.

It should be noted that, when converse modalities are added to the language (as in the basic tense logic), modal Sahlqvist formulas *can* be translated into pure $\mathcal{H}(@)$ formulas. And, indeed, in this case axiomatizations combining pure formulas and modal Sahlqvist formulas are always strongly complete for the relevant frame class [81,136].

There are a number of well known complete modal logics that cannot be axiomatized by means of Sahlqvist formulas, including PDL, GL and Grz. One might ask what happens when nominals and satisfaction operators are added to these logics. The following result, proved in [20,136], provides a partial answer. It shows that, under certain condition, a complete axiomatization of a modal logic can be turned into a complete axiomatization of the corresponding hybrid logic (in the language $\mathcal{H}(@)$). Recall that a modal logic *has a master modality* if there is a modality $[*]$ that satisfies the S4 axioms, such that $[*]p \to [R]p$ is derivable for all other modalities $[R]$ in the language (see also Chapter **??** and **??** of this handbook). Furthermore, recall the notion of *admitting filtration* defined in Chapter **??** of this handbook. Informally, a logic defined over a class of frames K admits filtration if each formula φ can be associated with a set of formulas Σ_φ (the "filtration set" of φ) such that for each model $\mathcal{M}$ based on a frame in K, and for each formula φ, there is a filtration of $\mathcal{M}$ over Σ_φ of which the underlying frame is in K.

Theorem 3.4 *Let Σ be any set of modal formulas such that the modal logic $\mathbf{K}+\Sigma$ is complete, admits filtration and has a master modality. Then $\mathbf{K}_{\mathcal{H}(@)} + \Sigma$ is also complete.*

GL, Grz and PDL all meet the requirements of Theorem 3.4. Incidentally, a similar transfer result cannot exist for $\mathcal{H}(@, \downarrow)$. Indeed, the $\mathcal{H}(@, \downarrow)$-logic of the frame class defined by GL (i.e., the class of transitive and conversely well-founded frames) is not recursively axiomatizable [136].

We now prove Theorem 3.2 using a technique similar to that used in a standard, Henkin-style completeness proof for first-order logic [58]. The general argument runs as follows: we will show that every consistent set of formulas can be extended to a maximal consistent set satisfying certain properties. Next, we will construct out of each such maximal consistent set a model, whose domain consists of equivalence classes of nominals. Finally, we show that the constructed model satisfies the original set of formulas, and that the underlying frame satisfies the relevant frame conditions.

The proof of the following lemma is straightforward.

Lemma 3.5 *The following formulas and rule are derivable in $\mathbf{K}_{\mathcal{H}(@)} + \Sigma$.*

[4] This essentially follows from the proof by substitutions of the Sahlqvist correspondence theorem (cf. Chapter **??** of this handbook), since the substitutions used only involve a bounded form of quantification. See Section 3.2 for more information on the tight relationship between bounded quantification and $\mathcal{H}(@, \downarrow)$.

(i) $\vdash @_j k \to (@_j\psi \leftrightarrow @_k\psi)$

(ii) $\vdash @_j(\psi_1 \wedge \psi_2) \leftrightarrow @_j\psi_1 \wedge @_j\psi_2$

(iii) $\vdash @_j\neg\psi \leftrightarrow \neg @_j\psi$

(iv) $\vdash @_j @_k\psi \leftrightarrow @_k\psi$

(v) $\vdash @_j\langle R\rangle k \wedge @_k\psi \to @_j\langle R\rangle\psi$

(vi) *If* $\vdash @_i\langle R\rangle j \wedge @_j\varphi \to \psi$ *then* $\vdash @_i\langle R\rangle\varphi \to \psi$*, provided* $i \neq j$ *and* j *does not occur in* φ *or* ψ*.*

We can now prove a Lindenbaum Lemma that shows how to extend any consistent set of formulas to a maximally consistent set, but in addition we will ensure that all diamonds are “witnessed” by nominals.

Lemma 3.6 *Every* $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$*-consistent set* Γ *can be extended to a maximal* $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$*-consistent set* Γ^+ *such that*

(i) *One of the elements of* Γ^+ *is a nominal;*

(ii) *For all* $@_i\langle R\rangle\varphi \in \Gamma$ *there is a nominal* j *such that* $@_i\langle R\rangle j \in \Gamma$ *and* $@_j\varphi \in \Gamma$*.*

Proof. By expanding the language with countably many nominals, we can ensure that there are infinitely many nominals that do not occur in Γ, while preserving consistency of Γ. Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of the nominals of the extended language, and let $(\varphi_n)_{n\in\mathbb{N}}$ be an enumeration of all $\mathcal{H}(@,\downarrow)$-formulas of the extended language. We will construct Γ^+ as the limit of an infinite sequence $\Gamma^0 \subseteq \Gamma^1 \subseteq \Gamma^2 \subseteq \cdots$.

Let Γ^0 denote $\Gamma \cup \{i\}$, for some nominal i not occurring in Γ. Then Γ_0 is consistent, for suppose not. Then there are $\varphi_1, \ldots, \varphi_n$ such that $\vdash_{\mathbf{K}_{\mathcal{H}(@,\downarrow)}+\Sigma} i \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. By the (Gen$_@$) rule and the (K$_@$) axiom, it follows that $\vdash_{\mathbf{K}_{\mathcal{H}(@,\downarrow)}+\Sigma} @_i i \to @_i\neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. By the (Ref$_@$) axiom and the (MP) rule, $\vdash_{\mathbf{K}_{\mathcal{H}(@,\downarrow)}+\Sigma} @_i\neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$, and hence, by the (Name) rule, $\vdash_{\mathbf{K}_{\mathcal{H}(@,\downarrow)}+\Sigma} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. But this contradicts the fact that Γ is consistent.

For $k \in \mathbb{N}$, define Γ^{k+1} as follows:

(i) $\Gamma^{k+1} = \Gamma^k$ if $\Gamma^k \cup \{\varphi_k\}$ is $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$-inconsistent,

(ii) otherwise
   (a) $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k\}$ if φ_k is not of the form $@_i\langle R\rangle\psi$.
   (b) $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k, @_i\langle R\rangle i_m, @_{i_m}\psi\}$ if φ_k is of the form $@_i\langle R\rangle\psi$, where i_m is the first nominal that does not occur in Γ^k or φ_k.

Each step preserves consistency: if Γ^k is $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$-consistent, then so is Γ^{k+1}. The only non-trivial case is (ii.b), and we will prove that also in this case, consistency is preserved.

Let $\Gamma^k \cup \{\varphi_k\}$ be $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$-consistent, let φ_k be of the form $@_i\langle R\rangle\psi$, and suppose for the sake of contradiction that $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k, @_i\langle R\rangle i_m, @_{i_m}\psi\}$ is not $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$-consistent. Then there are $\varphi_1, \ldots, \varphi_n \in \Gamma^k$ such that $\vdash_{\mathbf{K}_{\mathcal{H}(@,\downarrow)}+\Sigma} (\varphi_k \wedge @_i\langle R\rangle i_m \wedge @_{i_m}\psi) \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. It follows by the last clause of Lemma 3.5 that $\vdash_{\mathbf{K}_{\mathcal{H}(@,\downarrow)}+\Sigma} \varphi_k \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. But this contradicts the fact that $\Gamma^+ \cup \{\varphi_k\}$ is $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$-consistent. We conclude that Γ^k is consistent.

Since $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$-consistency is preserved at each stage, it follows that $\Gamma^+ = \bigcup_{n\in\mathbb{N}} \Gamma^n$ is $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$-consistent. It is easy to see that Γ^+ also satisfies the other requirements in Lemma 3.6. $\Box$

We can proceed with the proof of Theorem 3.2.

Proof of Theorem 3.2. We first treat the case of $\mathcal{H}(@,\downarrow)$. Let Γ be a $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$ consistent set of $\mathcal{H}(@,\downarrow)$-formulas and Γ^+ a maximal $\mathbf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$-consistent set of $\mathcal{H}(@,\downarrow)$-formulas extending Γ, satisfying the conditions of Lemma 3.6. For $i \in \mathsf{NOM}$, let $[i] = \{j \mid @_i j \in \Gamma^+\}$.

Define the hybrid model $\mathcal{M} = \langle W, (R^{\mathcal{M}})_{R\in\mathsf{REL}}, V\rangle$, where $W = \{[i] \mid i \text{ is a nominal occurring in } \Gamma^+\}$, $R^{\mathcal{M}} = \{([i],[j]) \mid @_i\langle R\rangle j \in \Gamma^+\}$, $V(p) = \{[i] \mid @_i p \in \Gamma^+\}$ and $V(i) = \{[i]\}$.

Now, for all $\mathcal{H}(@,\downarrow)$-formulas φ and nominals i, $\mathcal{M},[i] \models \varphi$ iff $@_i\varphi \in \Gamma^+$. This truth lemma can be proved by a straightforward induction on φ, using the properties of Γ^+ and Lemma 3.5. For the inductive step for formulas of the from $\downarrow x.\psi$, we use the fact that Γ^+ contains all substitution instances of the (DA) axiom.

It follows that $\mathcal{M}, [i] \models \Gamma^+$, for $i \in \Gamma^+$ (recall that one of the elements of Γ^+ is a nominal). Since $\mathcal{M}$ is a named model (i.e., every point is named by a nominal) and Γ^+ contains all substitution instances of elements of Σ, all formulas in Σ are valid on the underlying frame of $\mathcal{M}$. We conclude that Γ is satisfiable on the class of frames defined by Σ.

For $\mathbf{K}_{\mathcal{H}(@)} + \Sigma$, the same argument applies. Note that the (DA) axiom was only used in the truth lemma, for the inductive step for formulas of the form $\downarrow x.\varphi$. □

In the above completeness proof, the role of the non-orthodox rules (Name) and (BG) is to ensure the existence of a named model. Named models have played a crucial role in the development of the model theory of hybrid languages. As we commented in Section 2.2, they were already used by the Sofia school in their axiomatic investigations for combinatory PDL. They are closely related to the notion of a *discrete general frame*, and with the work of Venema [144] completeness for modal logics containing the difference operator D.

3.2 Expressive Power and Characterization

In this section, we investigate the expressive power of the hybrid languages $\mathcal{H}(@)$ and $\mathcal{H}(@,\downarrow)$, both on the level of models and on the level of frames, and we compare it to the basic modal language and the first-order correspondence language. For further details on the results discussed in this section see [8,136].

3.2.1 Correspondence language and standard translations

¿From the point of view of first-order logic, nominals are nothing more than constants: they designate elements of the domain of the model. The first-order correspondence language of hybrid logic is therefore most naturally defined as follows.

Definition 3.7 The *first-order correspondence language* for hybrid logic is the first-order language with equality that contains a unary predicate P for each propositional variable $p \in \mathsf{PROP}$, a binary relation symbol for each modality $R \in \mathsf{REL}$ and a constant for each nominal $i \in \mathsf{NOM}$.

Any hybrid model $\mathcal{M} = \langle M, (R^{\mathcal{M}})_{R \in \mathsf{REL}}, V\rangle$ can be regarded as a model for the first-order correspondence language. The accessibility relations $R^{\mathcal{M}}$ are used to interpret the binary relation symbols, unary predicates are interpreted as the subsets that V assigns to the corresponding propositional variables, and constants are interpreted as the worlds that the corresponding nominals name. In what follows, we will not distinguish between hybrid models and models for the first-order correspondence language, and we will use the notation $\mathcal{M} = \langle M, (R^{\mathcal{M}})_{R \in \mathsf{REL}}, V\rangle$ for both.

The Standard Translation from modal logic into the first-order correspondence language (cf. Chapter **??** of this handbook) can be extended to hybrid languages. The translation for the hybrid language $\mathcal{H}(\mathsf{E}, @, \downarrow)$ is given in Figure 3 (left column), where $s, t \in \mathsf{NOM} \cup \mathsf{SVAR}$, $p \in \mathsf{PROP}$, and $R \in \mathsf{REL}$. Here, we conveniently identify the state variables of hybrid logic with the variables of the first-order correspondence language.

Proposition 3.8 (ST preserves truth) *For all hybrid formulas φ, hybrid models $\mathcal{M}$, states $w \in M$ and assignments g, $\mathcal{M}, g, m \models \varphi$ iff $\mathcal{M}, g^x_m \models ST_x(\varphi)$, where x is a variable not occurring in φ.*

As it turns out, there is also a converse translation, mapping formulas of the first-order correspondence language to formulas of $\mathcal{H}(\mathsf{E}, @, \downarrow)$. It is given in the right column in Figure 3.

Proposition 3.9 (HT preserves truth) *Let φ be a formula of the first-order correspondence language. Then for every model $\mathcal{M}$, assignment g and for any state w, $\mathcal{M}, g^x_w \models \varphi$ iff $\mathcal{M}, g, w \models \downarrow x.HT(\varphi)$.*

It follows that $\mathcal{H}(\mathsf{E}, @, \downarrow)$ is as expressive as the first-order correspondence language. In fact, the satisfaction operators can be defined in terms of E (namely, $@_i\varphi$ is equivalent to $\mathsf{E}(i \wedge \varphi)$), and therefore $\mathcal{H}(\mathsf{E}, \downarrow)$ is already as expressive as the first-order correspondence language [5] . This leaves the question open of what is the range

[5] A similar translation can be given for $\mathcal{H}(@, \forall)$, see [32].

$ST_t(\top)$	=	$\top$			
$ST_t(s)$	=	$(t = s)$			
$ST_t(p)$	=	$P(t)$	$HT(\top)$	=	$\top$
$ST_t(\neg\varphi)$	=	$\neg ST_t(\varphi)$	$HT(R(s, s'))$	=	$@_s\langle R\rangle s'$
$ST_t(\varphi \wedge \psi)$	=	$ST_t(\varphi) \wedge ST_t(\psi)$	$HT(P(s))$	=	$@_s p$
$ST_t(\langle R\rangle\varphi)$	=	$\exists y.(R(t, y) \wedge ST_y(\varphi))$	$HT(s = t)$	=	$@_s t$
$ST_t(\mathsf{E}\varphi)$	=	$\exists y.ST_y(\varphi)$	$HT(\neg\varphi)$	=	$\neg HT(\varphi)$
$ST_t(@_s\varphi)$	=	$ST_s(\varphi)$	$HT(\varphi \wedge \psi)$	=	$HT(\varphi) \wedge HT(\psi)$
$ST_t(\downarrow z.\varphi)$	=	$\exists z.(z = t \wedge ST_t(\varphi))$	$HT(\exists x.\varphi)$	=	$\mathsf{E}{\downarrow} x.HT(\varphi)$

where y a variable distinct from the term t and not occurring in φ

Fig. 3. Standard Translation ST and Hybrid Translation HT

of ST for languages weaker than $\mathcal{H}(\mathsf{E}, @, \downarrow)$, i.e., which formulas of the first-order correspondence language are (equivalent to) translations of formulas of these hybrid languages? We will discuss this issue in the next section.

3.2.2 Characterizing expressivity on models

In this section, we address in detail the question of which formulas of the first-order correspondence language are equivalent to (standard translations of) hybrid formulas.

First, let us generalize the notion of bisimulation to hybrid languages.

Definition 3.10 Let $\mathcal{M} = \langle M, (R^{\mathcal{M}})_{R\in\mathsf{REL}}, V\rangle$ and $\mathcal{N} = \langle N, (S^{\mathcal{N}})_{S\in\mathsf{REL}}, U\rangle$ be hybrid models. A *hybrid bisimulation between $\mathcal{M}$ and $\mathcal{N}$* is a non-empty binary relation $Z \subseteq M \times N$ such that the following clauses hold

(atom) If $Z(m, n)$, then $m \in V(p)$ iff $n \in U(p)$, for $p \in \mathsf{PROP} \cup \mathsf{NOM}$.

(nom) If $V(i) = \{m\}$ and $U(i) = \{n\}$ then $Z(m, n)$, for $i \in \mathsf{NOM}$.

(forth) If $Z(m, n)$ and $R^{\mathcal{M}}(m, m')$, then there is an $n' \in N$ such that $S^{\mathcal{N}}(n, n')$ and $Z(m', n')$.

(back) If $Z(m, n)$ and $S^{\mathcal{N}}(n, n')$, then there is an $m' \in M$ such that $R^{\mathcal{M}}(m, m')$ and $Z(m', n')$.

A formula $\varphi(x_1, \ldots, x_n)$ of the first-order correspondence language is said to be *invariant for bisimulations* if for all bisimulations Z between hybrid models $\mathcal{M}$ and $\mathcal{N}$ and for all assignments g and h with $Z(g(x_k), h(x_k))$ for $k = 1 \ldots n$, it is the case that $\mathcal{M}, g \models \varphi$ iff $\mathcal{N}, h \models \varphi$.

Theorem 3.11 *A formula φ of the first-order correspondence language with at most one free variable x is equivalent to the standard translation of an $\mathcal{H}(@)$-formula iff φ is invariant under hybrid bisimulations.*

The proof is a straightforward generalization of the one for the basic modal language. As a corollary of Theorem 3.11, we obtain the following syntactic characterization.

Corollary 3.12 *A formula φ of the first-order correspondence language with at most one free variable x is equivalent to the standard translation of an $\mathcal{H}(@)$-formula iff φ is equivalent to a formula generated by the following recursive definition, where t is a term (constant or variable), c is a constant, and x is a variable distinct from t:*

$$\varphi ::= \top \mid P(t) \mid t = c \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x.(R(t, x) \wedge \varphi).$$

Proof. One direction of the claim follows from the fact that $ST(\varphi)$ is of the given form, for each $\mathcal{H}(@)$-formula φ. As for the other direction, a straightforward induction shows that every first-order formula of the given form is invariant under hybrid bisimulations, and hence every such formula with at most one free variable is equivalent to (the standard translation of) an $\mathcal{H}(@)$-formula. □

Let us now consider the language $\mathcal{H}(@, \downarrow)$. First, we will give a syntactic characterization (see [8] for details). Call a first-order formula *bounded* if it is built up from atomic formulas using the Boolean connectives and bounded quantification of the form $\exists x.(R(s,x) \wedge \cdot)$ or $\forall x.(R(s,x) \rightarrow \cdot)$, where s is a term distinct from the variable x.

Theorem 3.13 *A formula φ of the first-order correspondence language with one free variable is equivalent to the standard translation of a $\mathcal{H}(@, \downarrow)$ sentence iff φ is equivalent to a bounded formula.*

Proof. The standard translation of an $\mathcal{H}(@, \downarrow)$ sentence is always a bounded formula of the correspondence language. Conversely, we can extend the translation HT given in Figure 3 with the following clause for bounded quantification:

$$HT(\exists x.(R(s,x) \wedge \psi)) \;=\; @_s\langle R\rangle{\downarrow}x.HT(\psi).$$

In this way, we obtain, for each bounded formula φ of the first-order correspondence language, an $\mathcal{H}(@, \downarrow)$-formula $HT(\varphi)$. Moreover, a straightforward inductive argument shows that $HT(\varphi)$ is equivalent to φ, in the sense of Proposition 3.9. Recall that the formula φ in the statement of the Theorem contains at most one free variable x, and let φ' be any bounded formula equivalent to φ. It follows that φ' (and hence φ) is equivalent to $ST_x({\downarrow}x.HT(\varphi'))$. $\square$

In other words, $\mathcal{H}(@, \downarrow)$ corresponds to the bounded fragment of first-order logic. By means of the notion of generated submodels, we can semantically characterize this fragment.

Definition 3.14 Let $\mathcal{M} = \langle M, (R^{\mathcal{M}})_{R\in\mathsf{REL}}, V\rangle$ and $\mathcal{N} = \langle N, (R^{\mathcal{N}})_{R\in\mathsf{REL}}, V'\rangle$ be hybrid models. Then $\mathcal{N}$ is a *generated submodel* of $\mathcal{M}$ if $N \subseteq M$ and for all $w, v \in M$ and relation R_i, if $w \in N$ and $R_i(w,v)$ then $v \in N$, while R_i' and V' are the restrictions of R_i and V to N respectively. A formula φ is *invariant for generated submodels* if for all models $\mathcal{M}$, $\mathcal{N}$ such that $\mathcal{N}$ is a generated submodel of $\mathcal{M}$, and for all $\mathcal{N}$-assignments g, $\mathcal{M}, g \models \varphi$ if and only if $\mathcal{N}, g \models \varphi$.

Theorem 3.15 *A formula φ of the first-order correspondence language is invariant under generated submodels iff φ is equivalent to a bounded formula.*

Proof. Suppose a first-order formula φ is invariant under generated submodels. For convenience, we assume that φ is a sentence (free variables can be replaced by new constants). Let $c_1, \ldots, c_k$ be the constants and $R_1, \ldots, R_m$ be the binary relations occurring in φ, and let P be a new unary predicate. We will use $R(s,t)$ as a shorthand for $\bigvee_{1\leq i\leq m} R_i(s,t)$. Then the following holds:

$$\{\forall x.(R^n(c_l, x) \rightarrow P(x)) \mid 1 \leq l \leq k \text{ and } n \in \mathbb{N}\} \models \varphi \leftrightarrow \varphi^P,$$

where $R^n(x,y)$ is a shorthand for a bounded formula which expresses that y can be reached from x in exactly n steps along R (i.e., $\exists x_1(R(x,x_1) \wedge \exists x_2(R(x_1,x_2) \wedge \cdots (\cdots \wedge x_n = y)\cdots)))$ and φ^P is the result of relativising all quantifiers in φ by P (that is, $\exists x.\varphi$ becomes $\exists x.(P(x) \wedge \varphi)$ and $\forall x.\varphi$ becomes $\forall x.(P(x) \rightarrow \varphi)$). By compactness, it follows that there is an $m \in \mathbb{N}$ such that

$$\forall x.\Big(\Big(\bigvee_{1\leq l\leq k} R^{\leq m}(c_l, x)\Big) \rightarrow P(x)\Big) \models \varphi \leftrightarrow \varphi^P.$$

Let φ' be the result of relativising all quantifiers in φ by the formula $\big(\bigvee_{1\leq l\leq k}(R^{\leq m}(c_l,x))\big)$. It follows that $\models \varphi \leftrightarrow \varphi'$. Finally, φ' is (modulo some simple syntactic manipulations) a bounded sentence. $\square$

This result was first proved in the sixties by Feferman and Kreisel [61,59], and was independently proved by Areces, Blackburn and Marx [8] in the context of hybrid logic.

For any model $\mathcal{M}$ and world w, let $\mathcal{M}_w$ denote the smallest generated submodel of $\mathcal{M}$ containing w. In fact, it is easy to see that the domain of $\mathcal{M}$ contains precisely those worlds that are reachable in finitely many steps from w or from a world named by a nominal. As a corollary of the above results, we know that $\mathcal{M}, w$ and $\mathcal{M}_w, w$ agree on all sentences of $\mathcal{H}(@, \downarrow)$. If we combine this with the fact that all first-order formulas are invariant under potential isomorphisms, we obtain the following:

Proposition 3.16 *Let $\mathcal{M}$ and $\mathcal{N}$ be models, with corresponding states w, v. If there is a potential isomorphism between $\mathcal{M}_w$ and $\mathcal{N}_v$ connecting w to v, then $\mathcal{M}, w$ and $\mathcal{N}, v$ agree on all $\mathcal{H}(@, \downarrow)$-sentences.*

While the converse does not hold in general, it does hold on ω-saturated models. This means that "potential isomorphisms between point-generated submodels" capture $\mathcal{H}(@, \downarrow)$-indistinguishability in exactly the same way that potential isomorphisms capture first-order indistinguishability.

3.2.3 Characterizing frame definability

Given a set of hybrid formulas Σ, we say that the *frame class defined by* Σ is the class of frames in which every formula of Σ is valid. We say that a frame class is *elementary* (or *first-order definable*) if it is defined by a first-order sentence, in the language with equality and a relation symbol for each $R \in \mathsf{REL}$. The Goldblatt-Thomasson theorem tells us that an elementary frame class is definable by a set of formulas of the basic modal language iff the class is closed under disjoint unions, generated subframes, and bounded morphic images, and its complement is closed under ultrafilter extensions (see Chapter **??** of this handbook for this result and for a definition of the notions involved). In this section, we discuss analogues of this result for hybrid languages.

Due to the increased expressivity of hybrid languages, frame classes definable by hybrid formulas are in general not closed under disjoint unions or bounded morphic images. For example, the class of irreflexive frames, which is not closed under bounded morphic images, is defined in $\mathcal{H}(@)$ by the formula $i \to \neg\Diamond i$, and the class of frames that have exactly one element, which is not closed under disjoint unions, is defined by the formula i. Nevertheless, frame classes definable in $\mathcal{H}(@)$ are closed under generated subframes, and their complement is closed under ultrafilter extensions. In fact, a slightly stronger closure condition holds, involving a restricted form of bounded morphisms.

Definition 3.17 Let $\mathcal{F}$ and $\mathcal{G}$ be frames, and let $\mathfrak{ue}\mathcal{G}$ be an ultrafilter extension of $\mathcal{G}$. $\mathcal{G}$ is an *ultrafilter morphic image* of $\mathcal{F}$ if there is a surjective bounded morphism $f : \mathcal{F} \to \mathfrak{ue}\mathcal{G}$ such that $|f^{-1}(u)| = 1$ for all principal ultrafilters $u \in \mathfrak{ue}\mathcal{G}$.

Note first that whenever $\mathcal{G}$ is an ultrafilter morphic image of a frame $\mathcal{F}$, $\mathfrak{ue}\mathcal{G}$ is a bounded morphic image of $\mathcal{F}$. It follows that the validity of modal formulas is preserved under taking ultrafilter morphic images. Secondly, note that every frame is an ultrafilter morphic image of its ultrafilter extension. Hence, if a property of frames is preserved under ultrafilter morphic images, its complement is preserved under taking ultrafilter extensions.

Proposition 3.18 *All frame classes definable by a set of $\mathcal{H}(@)$-formulas are closed under taking ultrafilter morphic images.*

Proof. Let φ be an $\mathcal{H}(@)$-formula, let $f : \mathcal{F} \to \mathfrak{ue}\mathcal{G}$ be a surjective ultrafilter morphism, and suppose $\mathcal{G} \not\models \varphi$. We will show that $\mathcal{F} \not\models \varphi$.

Let V be a valuation and w a world such that $\langle \mathcal{G}, V\rangle, w \not\models \varphi$. Define the valuation $V^{\mathfrak{ue}}$ on $\mathfrak{ue}\mathcal{G}$ such that $V^{\mathfrak{ue}}(p) = \{u \mid V(p) \in u\}$ for all propositional variables p and $V^{\mathfrak{ue}}(i) = \{u \mid V(i) \in u\}$ for all nominals i. It is easily seen that $V^{\mathfrak{ue}}$ assigns to each nominal a singleton set consisting of a principal ultrafilter, and hence $V^{\mathfrak{ue}}$ is a well-defined hybrid valuation. Moreover, a standard argument [28, Proposition 2.59] shows that for all worlds v and formulas ψ, $\langle \mathcal{G}, V\rangle, v \models \psi$ iff $\langle \mathfrak{ue}\mathcal{G}, V^{\mathfrak{ue}}\rangle, \pi v \models \psi$, where πv is the principal ultrafilter generated by v. It follows that $\langle \mathfrak{ue}\mathcal{G}, V^{\mathfrak{ue}}\rangle, \pi w \not\models \varphi$.

Next, define the valuation V' for $\mathcal{F}$ such that $V'(p) = \{v \mid f(v) \in V^{\mathfrak{ue}}(p)\}$ for all propositional variables p and $V'(p) = \{v \mid f(v) \in V^{\mathfrak{ue}}(i)\}$ for all nominals i. Since f is injective on principal ultrafilters and nominals denote principal ultrafilters in $\mathfrak{ue}\mathcal{G}$, $V'(i)$ is a singleton for all nominals i, and hence $\langle \mathcal{F}, V'\rangle$ is a well-defined hybrid model. Furthermore, a standard argument shows that (the graph of) f is a hybrid bisimulation between $\mathfrak{ue}\mathcal{G}$ and $\mathcal{F}$. Since f is surjective, there is a $u \in \mathcal{F}$ such that $f(u) = \pi w$. By invariance under hybrid bisimulations, $\langle \mathcal{F}, V'\rangle, u \not\models \varphi$, and hence $\mathcal{F} \not\models \varphi$ □

We can strengthen Proposition 3.18 to the following characterization of frame definability in $\mathcal{H}(@)$ [136].

Theorem 3.19 *An elementary class of frames is definable by a set of $\mathcal{H}(@)$ formulas iff it is closed under taking ultrafilter morphic images and generated subframes.*

Proof. The easy direction is already discussed above: every frame class defined by a set of $\mathcal{H}(@)$-formulas is closed under taking ultrafilter morphic images and generated subframes. We will now prove the hard direction. Let K be any elementary frame class closed under taking ultrafilter morphic images and generated subframes, and let $Th(K)$ be the set of $\mathcal{H}(@)$-formulas valid on K. To show that K is $\mathcal{H}(@)$-definable, it suffices to show that $Th(K)$ itself defines K.

Suppose that $\mathcal{F} \models Th(K)$ for some frame $\mathcal{F}$ with domain W. For each subset $A \subseteq W$, introduce a propositional variable p_A, and for each $w \in W$, introduce a nominal i_w.[6] Let Δ be the set consisting of the following formulas, for all $A \subseteq W$, $v \in W$ and $R \in \mathsf{REL}$.

$$\begin{array}{ll} p_{-A} \leftrightarrow \neg p_A & \\ p_{A \cap B} \leftrightarrow p_A \wedge p_B & \\ p_{R^{-1}(A)} \leftrightarrow \langle R \rangle p_A & \quad \text{where } R^{-1}(A) = \{w \in W \mid \exists v \in A \text{ such that } wRv\} \\ i_v \leftrightarrow p_{\{v\}}. & \end{array}$$

Let $\Delta_{\mathcal{F}} = \{@_{i_v}[R_{i_1}] \cdots [R_{i_n}]\delta \mid v \in W, \delta \in \Delta, \text{ and } R_{i_1}, \ldots, R_{i_n} \in \mathsf{REL} \text{ with } n \in \mathbb{N}\}$. Intuitively, $\Delta_{\mathcal{F}}$ provides a full description of the frame $\mathcal{F}$. Clearly, $\Delta_{\mathcal{F}}$ is satisfiable on $\mathcal{F}$ under the natural valuation that sends p_A to A and i_v to $\{v\}$. We claim that $\Delta_{\mathcal{F}}$ is satisfiable on K. By compactness (recall that K is elementary), it suffices to show that every finite conjunction δ of elements of $\Delta_{\mathcal{F}}$ is satisfiable on K. But this follows immediately: δ is satisfiable on $\mathcal{F}$ and $\mathcal{F} \models Th(K)$, hence $\neg\delta \notin Th(K)$, i.e., δ is satisfiable on K.

Let $\langle \mathcal{G}, V \rangle \models \Delta_{\mathcal{F}}$ with $\mathcal{G} \in K$. Since K is closed under generated subframes, we may assume that $\mathcal{G}$ is generated by the set of points that are named by a nominal. It then follows that the model $\langle \mathcal{G}, V \rangle$ globally satisfies Δ. Let $\langle \mathcal{G}^*, V^* \rangle$ be an ω-saturated elementary extension of $\langle \mathcal{G}, V \rangle$ (such elementary extensions are known to exist even in the case of uncountable vocabularies). By elementarity, $\mathcal{G}^* \in K$ and $\langle \mathcal{G}^*, V^* \rangle$ globally satisfies Δ.

It can be shown that $\mathfrak{ue}\mathcal{F}$ is an ultrafilter morphic image of $\mathcal{G}^*$, where the ultrafilter morphism f is given by $f(v) = \{A \subseteq W \mid \langle \mathcal{G}^*, V^* \rangle, v \models p_A\}$. See [136] for further details. Since K is closed under ultrafilter morphic images, we conclude that $\mathcal{F} \in K$. $\square$

As we already discussed earlier, there is a particular interest in frame conditions definable by *pure* formulas, since these immediately yield complete axiomatizations. It would be worth having a characterization of the properties of frames that can be defined using pure formulas only. Details for such results can be found in [136], here we only state one theorem.

Definition 3.20 We say that a bisimulation Z between frames $\mathcal{F} = \langle F, (R^{\mathcal{F}})_{R \in \mathsf{REL}} \rangle$ and $\mathcal{G} = \langle G, (R^{\mathcal{G}})_{R \in \mathsf{REL}} \rangle$ *respects a set X of elements* of $\mathcal{G}$ if for all $x \in X$,

(i) $Z(w, x)$ and $Z(v, x)$ implies $w = v$, and
(ii) $Z(w, x)$ and $Z(w, v)$ implies $v = x$.

A *bisimulation system* from $\mathcal{F}$ to $\mathcal{G}$ is a function f that assigns to each finite subset $X \subseteq G$ a total bisimulation $f(X) \subseteq F \times G$ respecting X.

Theorem 3.21 *A class of frames is defined by a pure $\mathcal{H}(@)$-formula iff it is elementary and closed under taking images of bisimulation systems.*

An example of a frame condition that is not preserved under taking images of bisimulation systems is the Church-Rosser property.

Proposition 3.22 *The frame condition $\forall xyz.(R(x, y) \wedge R(x, z) \to \exists u.(R(y, u) \wedge R(z, u)))$ is not preserved under images of bisimulation systems.*

[6] Technically, this might involve adding uncountably many propositional variables and nominals to the language. However, this will not cause any problems below. Of course, individual formulas can only contain finitely many propositional variables and nominals.

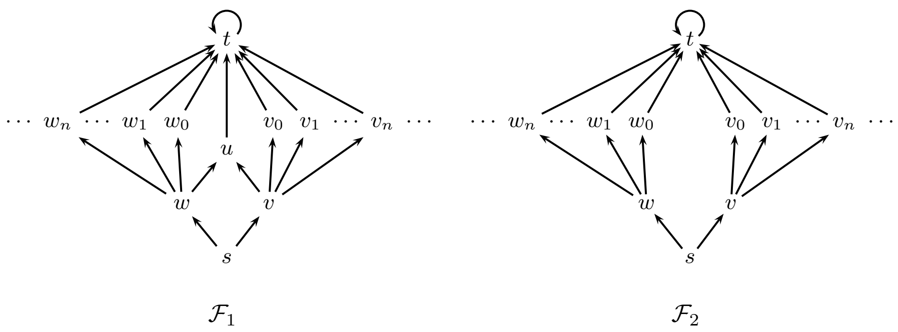

Fig. 4. Church-Rosser is not definable by pure formulas

Proof. Consider the two frames $\mathcal{F}_1 = \langle F_1, R^{\mathcal{F}_1}\rangle$ and $\mathcal{F}_2 = \langle F_2, R^{\mathcal{F}_2}\rangle$ shown in Figure 4. Notice that $\mathcal{F}_1$ is identical to $\mathcal{F}_2$, except for the additional point u (and its incoming and outgoing arrows). For any finite set $X \subseteq F_2$, let $f(X) = \{(w, w) \mid w \in F_1\} \cup \{(u, w_k), (u, v_l)\}$, for some $w_k, v_l \notin X$ (note that such w_k and v_l always exist). As is not hard to see, f is a bisimulation system. However, $\mathcal{F}_1$ satisfies the frame condition, while $\mathcal{F}_2$ does not. $\square$

It follows that the Church-Rosser property cannot be defined by pure formulas of $\mathcal{H}(@)$. A similar example is the class of transitive and atomic frames (where *atomicity* means that $\forall x.\exists y.(R(x, y) \wedge \forall z.(R(y, z) \to z = y)))$. This class of frames is defined by the modal formula $(\Diamond\Diamond p \to \Diamond p) \wedge (\Box\Diamond p \to \Diamond\Box p)$, but it cannot be defined by means of pure $\mathcal{H}(@)$-formulas, since it is not closed under images of bisimulation systems.

Finally, let us consider the language $\mathcal{H}(@, \downarrow)$. Interestingly, here the difference in frame definable power between pure formulas and arbitrary formulas is much smaller. In fact, every elementary frame property that can be defined by a set of $\mathcal{H}(@, \downarrow)$-sentences can already be defined by means of a single pure $\mathcal{H}(@, \downarrow)$-sentence. A precise characterization is given in the following theorem.

Definition 3.23 A frame $\mathcal{F}$ is a *finitely generated subframe* of a frame $\mathcal{G}$, if there is a finite set X of elements of the domain of $\mathcal{G}$, such that $\mathcal{F}$ is the submodel of $\mathcal{G}$ generated by X (i.e., such that $\mathcal{F}$ is the smallest generated submodel of $\mathcal{G}$ whose domain contains all elements of X).

We say that a frame class K *reflects finitely generated subframes* whenever it is the case for all frames $\mathcal{F}$ that, if every finitely generated subframe of $\mathcal{F}$ is in K, then $\mathcal{F} \in K$.

Theorem 3.24 *Let K be an elementary frame class. Then the following are equivalent:*

(i) *K is defined by a set of $\mathcal{H}(@, \downarrow)$ sentences.*
(ii) *K is defined by a single pure $\mathcal{H}(@, \downarrow)$ sentence.*
(iii) *K is closed under taking generated subframes, and reflects finitely generated subframes.*

This result can be extended to formulas containing only a limited number of nominals: let us say that a frame class K reflects n-point generated subframes whenever it is the case for all frames $\mathcal{F}$ that, if every subframe of $\mathcal{F}$ generated by at most n elements is in K, then $\mathcal{F} \in K$. Then Theorem 3.24 can be refined to the following result [8,136].

Theorem 3.25 *Let K be an elementary frame class and $n \in \mathbb{N}$. Then the following are equivalent:*

(i) *K is defined by a set of $\mathcal{H}(@, \downarrow)$ sentences containing (all together) at most n nominals.*
(ii) *K is defined by a single pure $\mathcal{H}(@, \downarrow)$ sentence containing at most n nominals.*
(iii) *K is closed under taking generated subframes, and reflects $(n + 1)$-generated subframes.*

Note that every modally definable frame class is closed under generated subframes and reflects point-generated subframes. It follows by the above result that every *elementary* modally definable frame class (in particular, every frame class defined by a modal Sahlqvist formula), is defined by a nominal-free pure sentence of $\mathcal{H}(@, \downarrow)$.

Language	*Frame classes defined by arbitrary formulas*	*Frame classes defined by pure formulas*
$\mathcal{H}$	closed under ultrafilter morphic images, and if every point-generated subframe of a frame $\mathcal{F}$ is a proper generated subframe of a frame in the class, then $\mathcal{F}$ is in the class	closed under images of bisimulation systems, and if every point-generated subframe of a frame $\mathcal{F}$ is a proper generated subframe of a frame in the class, then $\mathcal{F}$ is in the class
$\mathcal{H}(@)$	closed under ultrafilter morphic images and generated subframes	closed under images of bisimulation systems and generated subframes
$\mathcal{H}(\mathsf{E})$	closed under ultrafilter morphic images	closed under images of bisimulation systems
$\mathcal{H}(@, \downarrow)$	closed under generated subframes and reflecting finitely generated subframes.	closed under generated subframes and reflecting finitely generated subframes.

Fig. 5. Elementary frame classes definable in $\mathcal{H}$, $\mathcal{H}(@)$, $\mathcal{H}(\mathsf{E})$ and $\mathcal{H}(@, \downarrow)$

The most important results of this section are summarized in Figure 5 which also contains analogous results for the languages $\mathcal{H}$ and $\mathcal{H}(\mathsf{E})$. Again, full details can be found in [136].

3.3 Interpolation and Beth Definability

We will now turn to the properties of interpolation and Beth definability. The results in this section are mainly based on [8,136].

Recall that the modal logic of a class for frames K has interpolation if whenever $\varphi \to \psi$ is valid in K, then there exists a formula θ (called the *interpolant*) such that $\varphi \to \theta$ and $\theta \to \psi$ are valid in K, and all propositional variables occurring in θ occur both in φ and in ψ [7]. This definition can be generalized to hybrid logics in two ways, depending on whether only the propositional variables or also the nominals occurring in the interpolant are required to occur both in φ and in ψ. We will say that a hybrid logic has interpolation *over propositional variables* or *over propositional variables and nominals* to distinguish between these definitions.

The basic hybrid language $\mathcal{H}(@)$ lacks interpolation over nominals [8], as can be seen by the valid implication $i \wedge \Diamond i \to (j \to \Diamond j)$. An interpolant to this implication (which should express that the actual world is related to itself) is not allowed to contain any nominals. It is easily seen, using a bisimulation argument, that no such interpolant exists. Interpolation over proposition variables *does* hold. In fact, it holds relative to many frame classes [140,136]:

Theorem 3.26 *$\mathcal{H}(@)$ has interpolation over propositional variables relative to any frame class definable by a set of first-order universal Horn sentences.*

For $\mathcal{H}(@, \downarrow)$, we have better results: it has interpolation over proposition variables *and nominals* relative to many frame classes [8,136]:

Theorem 3.27 *$\mathcal{H}(@, \downarrow)$ has interpolation over propositional variables and nominals relative to any frame class definable by a set of nominal-free $\mathcal{H}(@, \downarrow)$ sentences. Moreover, $\mathcal{H}(@, \downarrow)$ has interpolation over proposition variables relative to any frame class definable by a set of $\mathcal{H}(@, \downarrow)$-sentences (possibly containing nominals).*

Theorem 3.27 covers many frame classes. Indeed, we saw in the previous section that every modally definable elementary frame class can be defined by a nominal-free sentence of $\mathcal{H}(@, \downarrow)$. It was shown in [30] that the interpolants can be effectively computed from a tableau proof (see also Section 5.3) [8]. The interpolation algorithm presented in [30] is conservative: on purely modal input it computes interpolants in which the hybrid syntactic machinery does not occur.

[7] This is sometimes called *local interpolation* or *arrow interpolation*, and in particular we are presenting it in its semantic version. We will not discuss global interpolation.

[8] Theorem 3.27 is related to a result by Feferman and Kreisel [61,59] who proved that the bounded fragment of first-order logic has interpolation by means of a cut free sequent calculus.

Given that $\mathcal{H}(@)$ lacks interpolation over nominals and $\mathcal{H}(@, \downarrow)$ has it, and given that $\mathcal{H}(@, \downarrow)$ has an undecidable satisfiability problem (as we will see in the next section), it is natural to ask whether there is any decidable hybrid language with interpolation over nominals. The answer is negative [135]: every extension of the minimal hybrid language $\mathcal{H}$ (satisfying certain regularity conditions such as allowing substitution of one nominal by another) either lacks interpolation or is undecidable. Moreover, $\mathcal{H}(@, \downarrow)$ is the least expressive extension of $\mathcal{H}(@)$ (satisfying the same regularity conditions) with interpolation over nominals.

The following can be seen as a weak version of this result. The proof is illustrative.

Theorem 3.28 *If $\mathcal{H}(@)$ has interpolation over nominals on a frame class K, then $\mathcal{H}(@)$ is expressively complete for $\mathcal{H}(@, \downarrow)$ on K (i.e., for each formula $\varphi \in \mathcal{H}(@, \downarrow)$, there exist a formula $\varphi' \in \mathcal{H}(@)$ such that φ and φ' are equivalent on K).*

Proof. Assume that $\mathcal{H}(@)$ has interpolation over nominals on K. We will show that every $\mathcal{H}(@, \downarrow)$ sentence φ is equivalent (on K) to an $\mathcal{H}(@)$-formula, proceeding by induction on the length of φ. The only interesting case here is when φ is of the form $\downarrow x.\psi(x)$. Let i and j be nominals not occurring in $\downarrow x.\psi(x)$. By induction, we know that $\psi(i)$ and $\psi(j)$ are equivalent to $\mathcal{H}(@)$-formulas $\psi'(i)$ and $\psi'(j)$ respectively. Now, the following implication is valid:

$$K \models i \wedge \psi'(i) \to (j \to \psi'(j)).$$

Any interpolant θ for this valid implication is equivalent to $\downarrow x.\psi(x)$. For, consider any model $\mathcal{M}$ and world w such that $\mathcal{M}, w \models \downarrow x.\psi(x)$. Let $\mathcal{M}[i/w]$ be the model that differs from $\mathcal{M}$ only in the fact that w is the denotation of i. Since i does not occur in $\downarrow x.\psi(x)$, we have that $\mathcal{M}[i/w], w \models \downarrow x.\psi(x)$, hence $\mathcal{M}[i/w], w \models i \wedge \psi(i)$. It follows that $\mathcal{M}[i/w], w \models \theta$. Since i does not occur in θ, it follows that $\mathcal{M}, w \models \theta$. Conversely, suppose $\mathcal{M}, w \models \theta$. Let $\mathcal{M}[j/w]$ be the model that differs from $\mathcal{M}$ only in the fact that j denotes w. Since j does not occur in θ, we have that $\mathcal{M}[j/w], w \models \theta$. It follows that $\mathcal{M}[j/w], w \models j \to \psi(j)$, and hence $\mathcal{M}[j/w], w \models \downarrow x.\psi(x)$. Since j does not occur in $\downarrow x.\psi(x)$, it follows that $\mathcal{M}, w \models \downarrow x.\psi(x)$. □

To conclude our discussion on interpolation, we consider the notion of *uniform interpolants*. As is discussed in Chapter **??** of this handbook, the modal logics K, S5, Grz and GL enjoy a very special form of interpolation, called uniform interpolation. For any formula φ, let $\mathsf{PROP}(\varphi)$ be the set of propositional variables occurring in φ. Then a modal logic has uniform interpolation if for every formula φ and for any $P \subseteq \mathsf{PROP}(\varphi)$, there is a formula φ_P (called a *uniform interpolant*) such that for any formula ψ, if $\mathsf{PROP}(\psi) \cap \mathsf{PROP}(\varphi) \subseteq P$ and $\varphi \to \psi$ is derivable, then $\varphi_P \to \psi$ is derivable.

We can generalize the definition to hybrid logics, and say that a hybrid logic has *uniform interpolation over propositional variables* if for every formula φ and for any $P \subseteq \mathsf{PROP}(\varphi)$, there is a formula φ_P such that for any formula ψ, if $\mathsf{PROP}(\psi) \cap \mathsf{PROP}(\varphi) \subseteq P$, and all nominals occurring in ψ occur in φ, then $\varphi \to \psi$ is valid iff $\varphi_P \to \psi$ is valid. Note the requirement imposed on nominals in this definition. It turns out that the $\mathcal{H}(@)$-logics of the frame classes corresponding to the modal logics K, S5, Grz and GL have uniform interpolation over propositional variables [20,136].

Finally, to close this section we turn to the Beth definability property. Recall that a logic is said to have the Beth Definability property if, intuitively, every implicit definition can be made explicit. More precisely, let $\Gamma(p)$ be any set of formulas containing the proposition variables p and possibly other propositional variables and nominals. $\Gamma(p)$ defines p implicitly if in all models in which both $\Gamma(p)$ and $\Gamma(p')$ are true at every state, also $p \leftrightarrow p'$ is true at every state (here, p' is a propositional variable not occurring in $\Gamma(p)$, and $\Gamma(p')$ is obtained from $\Gamma(p)$ by replacing all occurrences of p by p'). In other words, $\Gamma(p)$ defines p implicitly if $\Gamma(p) \cup \Gamma(p') \models^{glo} p \leftrightarrow p'$, where $\models^{glo}$ denotes global entailment. The Beth property states that whenever $\Gamma(p)$ defines p implicitly, there exists a formula θ in which p does not occur, such that $\Gamma(p) \models^{glo} p \leftrightarrow \theta$ [9]. Clearly, θ is an explicit definition of p, relative to the theory $\Gamma(p)$.

The Beth definability property for a logic is typically established as a corollary of the interpolation property for propositional variables. In particular, the following theorem can be shown using the above interpolation results.

[9] This is sometimes called the *global Beth property*. We will not discuss the local Beth property here.

Theorem 3.29 *$\mathcal{H}(@, \downarrow)$ has the Beth definability property relative to any frame class defined by a set of $\mathcal{H}(@, \downarrow)$ sentences, and $\mathcal{H}(@)$ has the Beth definability property relative to any frame class defined by a set of first-order universal Horn formulas.*

Surprisingly, the minimal hybrid language $\mathcal{H}$ lacks the Beth property relative to the class of all frames [20].

4 Decidability and Complexity

In this section, we will review the complexity of the satisfiability problem for various hybrid logics. First, let us consider the language $\mathcal{H}(@)$. We start with some good news: the satisfiability problem of $\mathcal{H}(@)$ is PSPACE-complete [6]. We provide a game based argument for the upper bound.

Theorem 4.1 *$\mathcal{H}(@)$-satisfiability on the class of all frames is* PSPACE*-complete.*

Proof. We only discuss the mono-modal case (the multi-modal case is a simple extension). The lower bound follows from the PSPACE-hardness of classical modal logic. We show the upper bound by defining, given a formula φ, the notion of a φ-game between two players. We will show that the existential player has a winning strategy for the φ-game iff φ is satisfiable. Moreover, every φ-game stops after at most as many rounds as the modal depth of φ and the information on the playing board is polynomial in the length of φ. This implies that a PSPACE algorithm exists. Fix a formula φ and let k be the number of different nominals appearing in φ. A φ-Hintikka set is a maximal consistent set of subformulas of φ. We denote the set of subformulas of φ by $\mathsf{SF}(\varphi)$. The φ-game is played as follows. There are two players, $\forall$belard (male) and $\exists$loise (female). She starts the game by playing a collection $\{X_0, \ldots, X_k\}$ of Hintikka sets and specifying a relation R on them. $\exists$loise loses immediately if one of the following conditions is false:

(i) X_0 contains φ, and all others X_l contain at least one nominal occurring in φ.

(ii) no nominal occurs in two different Hintikka sets.

(iii) for all X_l, for all $@_i\psi \in \mathsf{SF}(\varphi)$, $@_i\psi \in X_l$ iff $\{i, \psi\} \subseteq X_k$, for some k.

(iv) for all $\Diamond\psi \in \mathsf{SF}(\varphi)$, if $R(X_l, X_k)$ and $\Diamond\psi \notin X_l$, then $\psi \notin X_k$.

Now $\forall$belard may choose an X_l and a "defect-formula" $\Diamond\psi \in X_l$. $\exists$loise must respond with a Hintikka set Y such that

(i) $\psi \in Y$ and for all $\Diamond\theta \in \mathsf{SF}(\varphi)$, $\Diamond\theta \notin X_l$ implies that $\theta \notin Y$.

(ii) for all $@_i\psi \in \mathsf{SF}(\varphi)$, $@_i\psi \in Y$ iff $\{i, \psi\} \subseteq X_k$, for some k.

(iii) if $i \in Y$ for some nominal i, then Y is one of the Hintikka sets she played at the start. In this case the game stops and $\exists$loise wins.

If $\exists$loise cannot find a suitable Y, the game stops and $\forall$belard wins. If $\exists$loise does find a suitable Y (one that is not covered by the halting clause in item (iii) above) then Y is added to the list of played sets, and play continues. $\forall$belard must now choose a defect $\Diamond\psi$ from the last played Hintikka set with the following restriction: in round k he can only choose defects $\Diamond\psi$ such that the modal depth of $\Diamond\psi$ is less than or equal to the modal depth of φ minus k. $\exists$loise must respond as before. She wins if she can survive all his challenges (in other words, he loses if he reaches a situation where he cannot choose any more defects).

The φ-game stops after at most modal depth of φ many rounds. The information on the board is at any stage of the game polynomial in the length of φ. We claim that $\exists$loise has a winning strategy iff φ is satisfiable.

The right-to-left direction is clear: $\exists$loise has a winning strategy if φ is satisfiable, for she need simply play by reading the required Hintikka sets off the model. For the other direction, suppose $\exists$loise has a winning strategy for the φ-game. We create a model $\mathcal{M}$ for φ as follows. The domain M is built in steps by following her winning strategy. M_0 consists of her initial move $\{X_0, \ldots, X_n\}$. Suppose M_j is defined. Then M_{j+1} consists of a copy of those Hintikka sets she plays when using her winning strategy for each of $\forall$belard's possible moves played in the Hintikka sets from M_j (except when she plays a Hintikka set from her initial move, then of course we do not make a copy). Let M be the disjoint union of all M_j for j smaller than the modal depth of φ. Set $R(m, m')$ iff for all $\Diamond\psi \in \mathsf{SF}(\varphi)$, $\Diamond\psi \notin m \Rightarrow \psi \notin m'$ holds, and set $V(p) = \{m \in M \mid p \in m\}$. The rules of the game guarantee that nominals are interpreted as singletons.

We claim that the following truth-lemma holds. For all $m \in M$ which she plays in round j (i.e., $m \in M_j$), for all ψ of modal depth less than or equal to the modal depth of φ minus j, $\mathcal{M}, m \models \psi$ if and only if $\psi \in m$. We only discuss the case of $\Diamond$, if $\Diamond\psi \in m$, then $\forall$belard challenged this defect, so $\exists$loise could respond with an m' containing ψ. Since for all $\Diamond\psi \in \mathsf{SF}(\varphi)$, $\Diamond\psi \notin m \Rightarrow \psi \notin m'$ holds, we have $R(m, m')$ and by induction hypothesis $\mathcal{M}, m \models \Diamond\psi$. If $\Diamond\psi \notin m$ but $R(m, m')$ holds, then by our definition of R, $\psi \notin m'$, so again $\mathcal{M}, m \not\models \Diamond\psi$. Since $\exists$loise plays a Hintikka set containing φ in the first round, $\mathcal{M}$ satisfies φ. $\Box$

Since satisfiability of basic modal formulas on the class of all frames is already PSPACE-complete, we can conclude that, in this case, the addition of nominals does not increase the complexity of the satisfiability problem (up to a polynomial). This is not always the case:

Proposition 4.2 *$\mathcal{H}$-satisfiability on the class of symmetric frames is* EXPTIME*-complete.*

Proof. For any modal formula φ, let $\varphi' = i \wedge \Diamond\neg i \wedge \Box\Box\Diamond i \wedge \Box(\neg i \rightarrow \varphi^{\neg i})$, where i is any nominal and $\varphi^{\neg i}$ is obtained from φ by relativising all modalities with $\neg i$ (that is, $\Diamond\varphi$ becomes $\Diamond(\neg i \wedge \varphi)$ and $\Box\varphi$ becomes $\Box(\neg i \rightarrow \varphi)$). It can be easily seen that if φ' holds at a world w in a symmetric model $\mathcal{M}$ then φ holds globally in the submodel of $\mathcal{M}$ generated by w, minus the world w itself. Conversely, a symmetric model on which φ holds globally can easily be turned into a model for φ'. It follows that, on symmetric frames, φ' is satisfiable iff φ is globally satisfiable.

The global satisfiability problem for modal formulas on the class of symmetric frames is known to be EXPTIME-complete [50]. Hence, the satisfiability problem for $\mathcal{H}$ on the class of symmetric frames is EXPTIME-hard. That the problem is inside EXPTIME will follow from Theorem 4.7. $\Box$

Note that the proof uses only a single nominal. The satisfiability problem for the modal logic of the class of symmetric frames, **KB**, is only PSPACE-complete [50]. Hence, assuming PSPACE $\neq$ EXPTIME, adding a single nominal already makes the satisfiability problem more complex. A similar blowup holds for tense logic: the satisfiability problem of the basic temporal logic is PSPACE, but the addition of a single nominal moves the complexity to EXPTIME. The complexity drops though when considering linear or branching time models (to NP-complete in the first case and to PSPACE-complete in the second) [7].

Adding nominals can even result in logics that are undecidable and lack the finite model property, as was first observed in the context of description logics [100,103]. The example below is taken from [20]. Consider the bi-modal language with modalities $\langle R_1 \rangle$ and $\langle R_2 \rangle$, and let $\mathsf{KB23}$ be the frame class defined by the following modal Sahlqvist formulas:

$$\textstyle\bigwedge_{1 \leq k \leq 3} \langle R_1 \rangle p_k \rightarrow \bigvee_{1 \leq k < l \leq 3} \langle R_1 \rangle (p_k \wedge p_l) \qquad \text{(at most 2 } R_1\text{-successors)}$$

$$\textstyle\bigwedge_{1 \leq k \leq 4} \langle R_1 \rangle \langle R_1 \rangle p_k \rightarrow \bigvee_{1 \leq k < l \leq 4} \langle R_1 \rangle \langle R_1 \rangle (p_k \wedge p_l) \qquad \text{(at most 3 two-step } R_1\text{-successors)}$$

$$p \rightarrow [R_2]\langle R_2 \rangle p \qquad (R_2 \text{ is symmetric).}$$

Proposition 4.3 *The modal logic of* $\mathsf{KB23}$ *has the finite model property and is decidable.*

Proof. First, consider the mono-modal logic axiomatized by the first two axioms. This logic is complete for a class of frames that is closed under taking subframes, and it has the bounded width property: no point has more than two successors. It follows that this logic has the finite model property and is decidable. Second, consider the mono-modal logic given by the last axiom. This logic, which is complete for the class of symmetric frames, has the finite model property [49] and its satisfiability problem is complete for PSPACE [50]. Since decidability and the finite model property are preserved under taking fusions [70], the result follows. $\Box$

Proposition 4.4 *The $\mathcal{H}$-logic of* $\mathsf{KB23}$ *is undecidable and lacks the finite model property.*

Proof. For any mono-modal formula φ with modality $\langle R_1 \rangle$, let $\varphi^* = i \wedge \langle R_2 \rangle \neg i \wedge [R_2][R_1]\langle R_2 \rangle i \wedge [R_2](\neg i \rightarrow \varphi^{\neg i})$. Here again $\varphi^{\neg i}$ is obtained from φ by relativising all modalities with $\neg i$ as above. By the same argument

Nr. of successors	$\mathcal{H}(@, \downarrow)$	*First-order correspondence language*
$\kappa = 1$	NP-complete	NExpTime-complete
$\kappa = 2$	NP-complete	Non-elementary decidable
$3 \leq \kappa < \omega$	NExpTime-complete	Π^0_1-complete (co-r.e., not decidable)
$\kappa = \omega$	Σ^0_1-complete (r.e., not decidable)	Σ^1_1-complete (highly undecidable)
$\kappa > \omega$	Π^0_1-complete (co-r.e., not decidable)	Π^0_1-complete (co-r.e., not decidable)

Fig. 6. Complexity of the satisfiability problem on mono-modal models with bounded out-degree

as in the proof of Proposition 4.2, φ' is satisfiable on KB23 iff φ is globally satisfiable on the class of (mono-modal) frames in which each point has at most two successors and at most three two-step successors. Global satisfiability of modal formulas on the latter frame class is undecidable [133]. It follows that the $\mathcal{H}$-logic of KB23 is also undecidable, and hence, since it is recursively enumerable (as follows from the elementarity of KB23), it lacks the finite model property. □

Next, let us consider the language $\mathcal{H}(@, \downarrow)$. As was observed in [6], $\mathcal{H}(@, \downarrow)$ is a *conservative reduction class* of first-order logic. Following [39], we call a fragment of first-order logic a conservative reduction class if there is a computable translation τ mapping first-order formulas to formulas in the fragment, such that for all formulas α, $\tau(\alpha)$ is satisfiable iff α is, and $\tau(\alpha)$ has a finite model iff α has. Every conservative reduction class has an undecidable (in fact Π^0_1-complete) satisfiability problem, as well as an undecidable (in fact Σ^0_1-complete) finite satisfiability problem [39].

Theorem 4.5 *$\mathcal{H}(@, \downarrow)$ is a conservative reduction class.*

Proof. The class of first-order formulas with equality in a single binary relation is known to be a conservative reduction class [39]. Now, consider the following translation from this first-order language to $\mathcal{H}(@, \downarrow)$, where i is a fixed nominal:

$$\begin{aligned}
(R(x,y))^* &= @_x\langle R\rangle y \\
(x = y)^* &= @_x y \\
(\neg\varphi)^* &= \neg\varphi^* \\
(\varphi \wedge \psi)^* &= \varphi^* \wedge \psi^* \\
(\exists x.\varphi)^* &= @_i\langle R\rangle{\downarrow}x.(\varphi^*).
\end{aligned}$$

Clearly, $(\cdot)^*$ is a computable function. Moreover, a first-order sentence φ is satisfiable (in a finite model) iff φ^* is satisfiable (in a finite model). First, suppose $\mathcal{M} \models \varphi$. Let $\mathcal{M}'$ be the model obtained from $\mathcal{M}$ by adding a new state w, labelled with nominal i, extending the relation R such that $(w, v) \in R$ for all states v in the domain of M. Then, clearly, $\mathcal{M}' \models \varphi^*$. Moreover, $\mathcal{M}'$ is finite if M is. Conversely, suppose $\mathcal{M}, w \models \varphi^*$. Let v be the state in $\mathcal{M}$ labelled by the nominal i, and let $\mathcal{M}'$ be the submodel of $\mathcal{M}$ whose domain consists of all successors of v. Then, clearly, $\mathcal{M}' \models \alpha$. Moreover, $\mathcal{M}'$ is finite if $\mathcal{M}$ is. □

Even though the satisfiability problem for $\mathcal{H}(@, \downarrow)$ is undecidable, in certain cases $\mathcal{H}(@, \downarrow)$ is still computationally more attractive than the full first-order language. For instance, the satisfiability problem for $\mathcal{H}(@, \downarrow)$ becomes decidable if we restrict the out-degree of the nodes in the model [139]. Figure 6 lists the results for mono-modal formulas. Here, for a given κ, we consider the class of frames were every node has strictly less than κ successors. In particular, if $\kappa = \omega$, then each state can only have finitely many successors, and if $\kappa = 1$, the relation is the empty relation.

The undecidability of $\mathcal{H}(@, \downarrow)$ does not depend on the presence of nominals or propositional variables: even without these, the satisfiability problem is undecidable. Similarly, the undecidability does not depend on nested occurrences of $\downarrow$. One successful way to syntactically restrict the language in order to obtain decidability, is to restrict the interaction between $\downarrow$ and the modalities [105,139]. In particular, it was proved in [139] that decidability is regained when formulas of the form $\cdots\Box(\cdots{\downarrow}x.(\cdots\Box\cdots)\cdots)\cdots$ are excluded. In other words, the undecidability of $\mathcal{H}(@,\downarrow)$ is caused by formulas that, when put in negation normal form, contain a $\downarrow$-binder that is both in the scope of a box operator and that contains in its scope a box operator. This result was shown to be tight [139].

To round off this section, we will discuss two useful complexity result that can be used to prove upper bounds for the complexity of various hybrid logics: the *loosely $\forall$-bounded fragment with constants* and the *hybrid μ-calculus*.

Consider any first-order language not containing function symbols, but possibly containing constants. A formula of such a language is called *loosely $\forall$-guarded* if it is built up from possibly negated atomic formulas using conjunction, disjunction, existential quantification and loosely guarded universal quantification, i.e., universal quantification of the form $\forall \boldsymbol{x}(\varphi \rightarrow \psi)$, where $\boldsymbol{x}$ is a sequence of variables and φ is an atomic formula containing all free variables of ψ.

Theorem 4.6 ([83,138]) *The satisfiability problem for loosely $\forall$-guarded first-order formulas is* 2EXPTIME-*complete. It is* EXPTIME-*complete when there is a uniform bound on the number of variables occurring in the formula (but not necessarily on the number of constants).*

Many hybrid logics can be translated into the loosely $\forall$-guarded fragment using only a limited number of variables. For such logics, Theorem 4.6 provides an EXPTIME upper bound.

The hybrid μ-calculus [124] extends the modal μ-calculus (cf. Chapter **??** of this handbook) with nominals, converse operators and the universal modality. It expressively subsumes many propositional dynamic and temporal logics, such as (hybrid) PDL and CTL. Sattler and Vardi [124] showed by means of automata that the satisfiability problem for the hybrid μ-calculus is EXPTIME-complete. Beside the fact that this result singles out a very expressive hybrid language that is still decidable in EXPTIME, it is interesting because the proof is based on tree automata. For any formula φ, an automaton A_φ on infinite trees is given that accepts precisely the “tree models” of φ. Checking whether φ is satisfiable then reduces to solving the emptiness problem for A_φ. The catch, in the case of the hybrid μ-calculus, is that the standard tree model property fails for this language. The key idea in the proof is that a model of a hybrid μ-formula φ can be transformed into a forest by properly choosing points to witness diamond formulas. See [124] for details.

Below, we will give instead an alternative proof by providing a polynomial time satisfiability preserving translation of the full hybrid μ-calculus into its nominal-free fragment. But first, let us review the syntax and semantics of the language. The hybrid μ-calculus makes use of set variables, which we will write as $x, y, \ldots$, and which should not be confused with the state variables of hybrid languages such as $\mathcal{H}(@,\downarrow)$. The syntax is defined by the following inductive definition [10] :

$$\varphi ::= p \mid i \mid x \mid \neg\varphi \mid \varphi\wedge\psi \mid \langle R\rangle\varphi \mid \langle \bar{R}\rangle\varphi \mid \mathsf{E}\varphi \mid \mu x.\varphi,$$

where $p \in \mathsf{PROP}$, $i \in \mathsf{NOM}$, $R \in \mathsf{REL}$ and where x is a set variable occurring only positively in φ (i.e., under an even number of negation signs). Since the language contains set variables, the semantics is defined with the help of assignments. Here, an assignment will be a function g that assigns to each set variable a subset of the domain of the model. The semantics, then, is given by the following truth definition.

[10] Our notation is slightly different from the one used in [124].

$$\begin{array}{lll}
\mathcal{M}, g, w \models p & \text{iff} & w \in V(p) \text{ for } p \in \mathsf{PROP} \cup \mathsf{NOM} \\
\mathcal{M}, g, w \models x & \text{iff} & w \in g(x) \\
\mathcal{M}, g, w \models \neg\varphi & \text{iff} & \mathcal{M}, g, w \not\models \varphi \\
\mathcal{M}, g, w \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, g, w \models \varphi \text{ and } \mathcal{M}, g, w \models \psi \\
\mathcal{M}, g, w \models \langle R \rangle\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } R(w, v) \text{ and } \mathcal{M}, g, v \models \varphi \\
\mathcal{M}, g, w \models \langle \bar{R} \rangle\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } R(v, w) \text{ and } \mathcal{M}, g, v \models \varphi \\
\mathcal{M}, g, w \models \mathsf{E}\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } \mathcal{M}, g, v \models \varphi \\
\mathcal{M}, g, w \models \mu x.\varphi & \text{iff} & \text{for all } W' \subseteq W, \text{ if } \{v \in W \mid \mathcal{M}, g^x_{W'}, v \models \varphi\} \subseteq W' \text{ then } w \in W'.
\end{array}$$

Theorem 4.7 ([124]) *The satisfiability problem for the hybrid μ-calculus is* ExpTime*-complete.*

Proof. We define a polynomial time satisfiability preserving translation from the full hybrid μ-calculus to its nominal-free fragment, i.e., the modal μ-calculus with converse operators and the existential modality. Since the latter language is ExpTime complete [143,48], the result follows.

Consider any formula φ of the hybrid μ-calculus containing nominals $i_1, \ldots, i_n$. For each nominal i_k, introduce a new distinct propositional variable q_k. In the translation we will define, each nominal will be uniformly replaced by the corresponding propositional variable. Clearly, we cannot force these propositional variables to denote singleton sets. We *can*, however, ensure that the formula in question does not distinguish between states named by the same nominal. To this end, we will use $\langle \equiv \rangle\psi$ as a shorthand for the formula $\mu x.(\psi \vee \bigvee_{k \leq n}(q_k \wedge \mathsf{E}(q_k \wedge x)))$, which says that ψ holds either at the current state, or at a state satisfying the same nominal as the current state, or in general at any state reachable from the current world in finitely many steps along the "satisfies the same nominal" relation. Now, define φ^* inductively, as follows:

$$\begin{aligned}
(i_k)^* &= \langle \equiv \rangle q_k \\
p^* &= \langle \equiv \rangle p \\
x^* &= \langle \equiv \rangle x \\
(\neg\psi)^* &= \neg\psi^* \\
(\psi \wedge \chi)^* &= \psi^* \wedge \chi^* \\
(\langle R \rangle\psi)^* &= \langle \equiv \rangle\langle R \rangle\langle \equiv \rangle\psi \\
(\langle \bar{R} \rangle\psi)^* &= \langle \equiv \rangle\langle \bar{R} \rangle\langle \equiv \rangle\psi \\
(\mathsf{E}\psi)^* &= \mathsf{E}\psi^* \\
(\mu x.\psi)^* &= \mu x.\psi^*.
\end{aligned}$$

Finally, let $\varphi^+ = \varphi^* \wedge \bigwedge_{k \leq n} \mathsf{E}p_k$. Note that φ^+ does not contain any nominals, and is only polynomially longer than φ. We will now show that φ and φ^+ are equi-satisfiable. One direction is trivial: if $\mathcal{M}, w \models \varphi$, then, assigning to each q_k the same (singleton) denotation as i_k, we obtain that $\mathcal{M}, w \models \varphi^+$ (note that, in this case, $\equiv$ is the identity relation). Conversely, suppose $\mathcal{M}, w \models \varphi^+$, with $\mathcal{M} = \langle W, (R^{\mathcal{M}})_{R \in \mathsf{REL}}, V \rangle$. Let $\equiv$ be the smallest equivalence relation on W such that $v \equiv u$ whenever v and u both satisfy q_k for some $k \leq n$. Let $\widehat{\mathcal{M}} = \langle W/_{\equiv}, (R^{\widehat{\mathcal{M}}})_{R \in \mathsf{REL}}, V' \rangle$, where $W/_{\equiv}$ is the set of $\equiv$-equivalence classes of W,

$$\begin{array}{lll}
R^{\widehat{\mathcal{M}}}([v], [u]) & \text{iff} & \text{there are } v' \in [v] \text{ and } u' \in [u] \text{ such that } R^{\mathcal{M}}(v', u'), \\
[v] \in V'(p) & \text{iff} & \text{there is a } v' \in [v] \text{ such that } v' \in V(p), \text{ and} \\
[v] \in V'(i_k) & \text{iff} & \text{there is a } v' \in [v] \text{ such that } v' \in V(q_k).
\end{array}$$

By construction, $V'(i_k)$ is a singleton set for each $k \leq n$. Moreover, it follows directly from the definition of

$$
\begin{array}{cc}
(I) & \dfrac{}{\varphi, \Gamma \vdash \Delta, \varphi} \\[2ex]
(N_L) & \dfrac{a, b, \Gamma[a] \vdash \Delta[a]}{a, b, \Gamma[b] \vdash \Delta[b]}
\end{array}
$$

$$
\begin{array}{cccc}
(\neg_L) & \dfrac{\Gamma \vdash \Delta, \varphi}{\neg\varphi, \Gamma \vdash \Delta} & (\neg_R) & \dfrac{\varphi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\varphi} \\[2ex]
(\vee_L) & \dfrac{\varphi, \Gamma \vdash \Delta \quad \psi, \Gamma \vdash \Delta}{(\varphi \vee \psi), \Gamma \vdash \Delta} & (\vee_R) & \dfrac{\Gamma \vdash \Delta, \varphi, \psi}{\Gamma \vdash \Delta, (\varphi \vee \psi)} \\[2ex]
(\langle r \rangle_L)^1 & \dfrac{\langle r \rangle a, @_a\varphi, \Gamma \vdash \Gamma}{\langle r \rangle\varphi, \Gamma \vdash \Delta} & (\langle r \rangle_R) & \dfrac{\Gamma \vdash \Delta, @_a\varphi \quad \Gamma \vdash \Delta, \langle r \rangle a}{\Gamma \vdash \Delta, \langle r \rangle\varphi} \\[2ex]
(\downarrow_L) & \dfrac{a, \varphi[x/a], \Gamma \vdash \Delta}{a, \downarrow x.\varphi, \Gamma \vdash \Delta} & (\downarrow_R) & \dfrac{a, \Gamma \vdash \Delta, \varphi[x/a]}{a, \Gamma \vdash \Delta, \downarrow x.\varphi} \\[2ex]
({}^{\vee}@_L) & \dfrac{a, \varphi, \Gamma \vdash \Delta}{a, @_a\varphi, \Gamma \vdash \Delta} & ({}^{\vee}@_R) & \dfrac{a, \Gamma \vdash \Delta, \varphi}{a, \Gamma \vdash \Delta, @_a\varphi} \\[2ex]
({}^{\wedge}@_L) & \dfrac{a, @_a\varphi, \Gamma \vdash \Delta}{a, \varphi, \Gamma \vdash \Delta} & ({}^{\wedge}@_R) & \dfrac{a, \Gamma \vdash \Delta, @_a\varphi}{a, \Gamma \vdash \Delta, \varphi}
\end{array}
$$

$$
(\text{name})^2 \; \dfrac{a, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad (\text{term})^3 \; \dfrac{a, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad (\text{term}^-)^3 \; \dfrac{\Gamma \vdash \Delta}{a, \Gamma \vdash \Delta}
$$

Restrictions:

1 if a does not occur in φ, Γ, Δ.

2 if a does not occur in Γ, Δ.

3 if all formulas in Γ, Δ are @-prefixed.

Fig. 7. Rules for the Sequent Calculus $\mathbf{S}_{\mathcal{H}(@,\downarrow)}$

$(\cdot)^*$ and $\widehat{\mathcal{M}}$ that, for all formulas ψ of the hybrid μ-calculus, and for all worlds $v \in W$,

$$\widehat{\mathcal{M}}, [v] \models \psi \quad \text{iff} \quad \mathcal{M}, v \models \psi^*$$

In particular, $\widehat{\mathcal{M}}, [w] \models \varphi$. $\square$

5 Proof Theory

In this section we discuss proof methods for hybrid logics, and show examples of how to use them. We will first present two "classical" proof systems (a sequent calculus and a natural deduction calculus), and then two others (a tableau calculus and a resolution calculus), which are usually considered more suitable for implementations. We will focus on the languages $\mathcal{H}(@)$ and $\mathcal{H}(@, \downarrow)$.

5.1 Sequent Calculus

The first modern results on proof theory for hybrid logics can be found in the work of Seligman in the area of Situation Theory [128,129]. This work deals with strong ($\forall$-based) systems, but many of the key ideas underlying hybrid deduction (in particular, the deductive significance of @) were first explored in these papers.

The calculus $\mathbf{S}_{\mathcal{H}(@,\downarrow)}$ in Figure 7 is from [130] where a sound and complete sequent calculus for hybrid logics is developed from a sequent calculus for first-order logics by a series of transformations. In the figure, $\Gamma \vdash \Delta$ is a sequent where Γ and Δ are sets of hybrid formulas and φ, Γ is taken to be $\{\varphi\} \cup \Gamma$. The techniques used are quite general and can be applied to a wide range of hybrid and modal logics. Notice that the calculus is cut free. It can be proved that the cut rule is admissible.

An interesting feature of $\mathbf{S}_{\mathcal{H}(@,\downarrow)}$ is that the calculus is not restricted to @-formulas, as the other calculus we are going to discuss in the following sections. Intuitively, an @-prefixed sequent calculus can be generalized to deal with all formulas by using nominals as follows. A single nominal a on the left side of a sequent is enough to anchor all non @-prefixed formulas to the same element and so removes the need for them to share an @ prefix. The price to pay for this is that the calculus does not have a subformula property, as a proof may contain any number of @-prefixes which are not present in the end sequent (introduced using the ${}^{\wedge}@$ rules). But it is easy to prove that only "one layer" of prefixes is needed in any proof, and define a version of the subformula property that takes this into account.

The presence of nominals and the @ operator in the calculus above is crucial. When the underlying modal logic is temporal logic, more flexibility is possible: Demri [57] presents a sequent system for nominal tense logic, which does not contains $@$.

Example 5.1 We prove the sequent $\downarrow x.\langle R\rangle(x \wedge p) \vdash p$ in $\mathbf{S}_{\mathcal{H}(@,\downarrow)}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b, @_a\langle R\rangle b, a, p \vdash p}\,(I)}{b, @_a\langle R\rangle b, a, p \vdash @_a p}\,({}^{\vee}@_R)}{b, @_a\langle R\rangle b, a \wedge p \vdash @_a p}\,(\neg_L),(\neg_R),\text{ and }(\vee_R)}{b, @_a\langle R\rangle b, @_b(a \wedge p) \vdash @_a p}\,({}^{\vee}@_L)}{@_a\langle R\rangle b, @_b(a \wedge p) \vdash @_a p}\,(\text{term})}{a, @_a\langle R\rangle b, @_b(a \wedge p) \vdash @_a p}\,(\text{term}^-)}{a, \langle R\rangle b, @_b(a \wedge p) \vdash p}\,({}^{\wedge}@_L)\text{ and }({}^{\wedge}@_R)}{a, \langle R\rangle(a \wedge p) \vdash p}\,(\langle R\rangle_L)}{a, \downarrow x.\langle R\rangle(x \wedge p) \vdash p}\,(\downarrow_L)}{\downarrow x.\langle R\rangle(x \wedge p) \vdash p}\,(\text{name})$$

5.2 Natural Deduction Calculus

Seligman proposed also a natural deduction system (again, not restricted to @-formulas) in [129]. However, the paper only proves soundness and completeness and does not discuss whether the calculus is normalizing. Braüner introduced in [42] an @-prefixed natural deduction calculus for $\mathcal{H}(@,\downarrow,\forall)$ and its sublanguages and established normalization. Figure 8 shows the rules corresponding to the $\mathcal{H}(@,\downarrow)$ fragment.

The system $\mathbf{ND}_{\mathcal{H}(@,\downarrow)}$ can be extended in a complete way with additional inference rules corresponding to first-order conditions on the accessibility relations expressed by *geometric theories* [11] . And as we said, the system $\mathbf{ND}_{\mathcal{H}(@,\downarrow)}$ enjoys normalization (even when extended with rules for geometric theories), and a suitable version of the subformula property that takes into account the use of @-formulas. See [42], for further details. In [43] Braüner compares his system with Seligman's (actually, a slight variation of Seligman's to ensure closure under substitutions), providing translations of proofs in both directions. These translations allows us to transfer reduction rules between Braüner's and Seligman's calculus, but they are not sufficient to ensure normalization of the latter. Hence, the normalization problem for Seligman's calculus is still open.

Example 5.2 We prove that $\downarrow x.\langle R\rangle(x \wedge p) \to p$ is a tautology in $\mathbf{ND}_{\mathcal{H}(@,\downarrow)}$:

[11] A first-order formula is geometric if it is built out of atomic formulas of the form $R(x, y)$ and $x = y$ using only the connectives $\perp$, $\wedge$, $\vee$ and $\exists$. A geometric theory is a finite set of closed first-order formulas each having the form $\forall \bar{x}(\varphi \to \psi)$ where the formulas φ and ψ are geometric.

$$(\wedge_I)\ \frac{@_a\varphi \quad @_a\psi}{@_a(\varphi\wedge\psi)} \qquad\qquad (\wedge_{E1})\ \frac{@_a(\varphi\wedge\psi)}{@_a\varphi} \qquad (\wedge_{E2})\ \frac{@_a(\varphi\wedge\psi)}{@_a\psi}$$

$$(\to_I)\ \frac{\begin{array}{c}[@_a\varphi]\\ \vdots\\ @_a\psi\end{array}}{@_a(\varphi\to\psi)} \qquad\qquad (\to_E)\ \frac{@_a(\varphi\to\psi) \quad @_a\varphi}{@_a\psi}$$

$$(\bot_1)^1\ \frac{\begin{array}{c}[@_a\neg\varphi]\\ \vdots\\ @_a\bot\end{array}}{@_a\varphi} \qquad\qquad (\bot_2)\ \frac{@_a\bot}{@_c\bot}$$

$$(@_I)\ \frac{@_a\varphi}{@_c@_a\varphi} \qquad\qquad (@_E)\ \frac{@_c@_a\varphi}{@_a\varphi}$$

$$([r]_I)^2\ \frac{\begin{array}{c}[@_a\langle r\rangle c]\\ \vdots\\ @_c\varphi\end{array}}{@_a[r]\varphi} \qquad\qquad ([r]_E)\ \frac{@_a[r]\varphi \quad @_a\langle r\rangle c}{@_c\varphi}$$

$$(\downarrow_I)^3\ \frac{\begin{array}{c}[@_a c]\\ \vdots\\ @_c\varphi[x/c]\end{array}}{@_a\downarrow x.\varphi} \qquad\qquad (\downarrow_E)\ \frac{@_a\downarrow x.\varphi \quad @_a c}{@_c\varphi[x/c]}$$

$$(\mathrm{Ref})\ \frac{}{@_a a} \qquad\qquad (\mathrm{Nom}_1)^4\ \frac{@_a c \quad @_a\varphi}{@_c\varphi} \qquad\qquad (\mathrm{Nom}_2)\ \frac{@_a c \quad @_a\langle r\rangle b}{@_c\langle r\rangle b}$$

Restrictions:

[1] φ is a propositional variable.

[2] c is not free in $@_a[r]\varphi$ or in any undischarged assumptions other than the specified occurrences of $@_a\langle r\rangle c$.

[3] c is not free in $@_a\downarrow x.\varphi$ or in any undischarged assumptions other than the specified occurrences of $@_a c$.

[4] φ is a propositional variable or a nominal.

Fig. 8. Rules for the natural deduction calculus $\mathbf{ND}_{\mathcal{H}(@,\downarrow)}$

$$\frac{\dfrac{\dfrac{[@_y(\downarrow x.\langle R\rangle(x\wedge p))]^1 \quad \dfrac{}{@_y y}(\mathrm{Ref})}{@_y\langle R\rangle(y\wedge p)}(\downarrow_E) \quad \dfrac{\dfrac{\dfrac{\dfrac{[@_y\neg p]^2 \quad \dfrac{\dfrac{[@_z(y\wedge p)]^3}{@_z y}(\wedge_{E1}) \quad \dfrac{[@_z(y\wedge p)]^3}{@_z p}(\wedge_{E2})}{@_y p}(\mathrm{Nom}_1)}{@_y\bot}(\to_E)}{@_z\bot}(\bot_2)}{@_z\neg(y\wedge p)}(\to_I)^3}{@_y[R]\neg(y\wedge p)}([R]_E)}{\dfrac{@_y\bot}{@_y p}(\bot_1)^2}(\to_E)}{@_y((\downarrow x.\langle R\rangle(x\wedge p))\to p)}(\to_I)^1$$

5.3 *Tableau Calculus*

In Figure 9 the rules for a tableau calculus for $\mathcal{H}(@,\downarrow)$ are given. This calculus was introduced in [29], where tableau calculi for a family of *quantified* hybrid logics are presented (these are extensions of the propositional

Constant Rules: $(\neg\bot)\ \dfrac{\neg @_s \bot}{\top}$ $\quad (\neg\top)\ \dfrac{\neg @_s \top}{\bot}$

Negation Rules: $(@)\ \dfrac{@_s \neg\varphi}{\neg @_s \varphi}$ $\quad (\neg @)\ \dfrac{\neg @_s \neg\varphi}{@_s \varphi}$

Conjunctive rules: $(\wedge)\ \dfrac{@_s(\varphi \wedge \vee)}{\begin{array}{c} @_s\varphi \\ @_s\psi \end{array}}$ $\quad (\neg\vee)\ \dfrac{\neg @_s(\varphi \vee \psi)}{\begin{array}{c} \neg @_s\varphi \\ \neg @_s\psi \end{array}}$ $\quad (\neg \rightarrow)\ \dfrac{\neg @_s(\varphi \rightarrow \psi)}{\begin{array}{c} @_s\varphi \\ \neg @_s\psi \end{array}}$

Disjunctive rules: $(\vee)\ \dfrac{@_s(\varphi \vee \psi)}{@_s\varphi \mid @_s\psi}$ $\quad (\neg\wedge)\ \dfrac{\neg @_s(\varphi \wedge \psi)}{\neg @_s\varphi \mid \neg @_s\psi}$ $\quad (\rightarrow)\ \dfrac{@_s(\varphi \rightarrow \psi)}{\neg @_s\varphi \mid @_s\psi}$

Diamond Rules: $(\langle r\rangle)\ \dfrac{@_s\langle r\rangle\varphi}{\begin{array}{c} @_s\langle r\rangle t \\ @_t\varphi \end{array}}$ for t new in branch $\quad (\neg[r])\ \dfrac{\neg @_s[r]\varphi}{\begin{array}{c} @_s\langle r\rangle t \\ \neg @_t\varphi \end{array}}$ for t new in branch

Box Rules: $([r])\ \dfrac{@_s[r]\varphi \quad @_s\langle r\rangle t}{@_t\varphi}$ $\quad (\neg\langle r\rangle)\ \dfrac{\neg @_s\langle r\rangle\varphi \quad @_s\langle r\rangle t}{\neg @_t\varphi}$

@ rules: $(@)\ \dfrac{@_s @_t\varphi}{@_t\varphi}$ $\quad (\neg @)\ \dfrac{\neg @_s @_t\varphi}{\neg @_t\varphi}$

(Ref) $\dfrac{[s \text{ on the branch}]}{@_s s}$ $\quad$ (Nom) $\dfrac{@_s t \quad @_s\varphi}{@_t\varphi}$ $\quad$ (Bridge) $\dfrac{@_s t \quad @_u\langle r\rangle s}{@_u\langle r\rangle t}$

Downarrow Rules: $(\downarrow)\ \dfrac{@_s{\downarrow}x.\varphi}{@_s\varphi[s/x]}$ $\quad (\neg\downarrow)\ \dfrac{\neg @_s{\downarrow}x.\varphi}{\neg @_s\varphi[s/x]}$

Fig. 9. Rules for the tableau calculus $\mathbf{T}_{\mathcal{H}(@,\downarrow)}$

calculus defined in [25]). As in the case of natural deduction, the calculus is @-based: to prove the unsatisfiability of φ, apply the rules in Figure 9 to $@_i\varphi$ for i a nominal not in φ. If a closed tableau is found (i.e., a tableau in which each branch contains a pair of formulas $@_j\psi$ and $@_j\neg\psi$), then the original formula is unsatisfiable.

Completeness of the tableau calculus is proved for frame classes that can be axiomatized by pure, nominal free hybrid sentences [12] . Moreover, the tableau calculus can be used for effectively computing interpolants for a pair of formulas φ, ψ such that $\varphi \rightarrow \psi$ is a validity. The following result is proved in [30] using Fitting's argument for proving the same property for first-order logic [64].

Theorem 5.3 *Given a closed hybrid tableau for $\varphi \rightarrow \psi$ using the rules of $\mathbf{T}_{\mathcal{H}(@,\downarrow)}$, the interpolant can be computed effectively.*

In a slightly different direction, Tzakova [142] presents a general approach to hybrid tableaux using Fitting-style prefix calculi. Such tableau use nominals both as part of the object language and as meta-logical labels.

Tableau methods have played a crucial role in modern automated reasoning for modal logics, and the best state-of-the-art provers for modal-like logics (such as the description logics provers RACER [85,84] or FACT [90,89]) are based on tableaux (see Chapter **??** of this handbook for further details). A variation of the hybrid tableau calculus of Figure 9 has been equipped with heuristics to ensure termination in [38]. The ideas used are related to the techniques used for terminating tableaux for the description logic $\mathcal{SHOIQ}$ [92].

Example 5.4 We prove that ${\downarrow}x.(\langle R\rangle(x \wedge p) \rightarrow p)$ is a tautology in $\mathbf{T}_{\mathcal{H}(@,\downarrow)}$:

[12] The completeness proof is interesting: a valid hybrid sentence is translated into a valid first-order sentence in the correspondence language for which first-order closed tableau should exist; the tableau proof is then translated back into a hybrid tableau proof.

1.	$\neg @_i(\downarrow x.(\langle R\rangle(x \wedge p) \rightarrow p))$	Negation of the input formula
2.	$\neg @_i(\langle R\rangle(i \wedge p) \rightarrow p)$	$(\neg\downarrow)$ in 1
3.	$@_i(\langle R\rangle(i \wedge p)$	$(\neg \rightarrow)$ in 2
4.	$\neg @_i p$	$(\neg \rightarrow)$ in 2
5.	$@_i\langle R\rangle j$	$(\langle R\rangle)$ in 3
6.	$@_j(i \wedge p)$	$(\langle R\rangle)$ in 3
7.	$@_j i$	$(\wedge)$ in 6
8.	$@_j p$	$(\wedge)$ in 6
9.	$@_i p$	(Nom) in 7 and 8
	$\times$	Clash between 4 and 9

5.4 Resolution Calculus

As we just mentioned, the most successful automated theorem proving implementations for modal logics are based on the tableau method. Much of their outstanding performance is due to the heavy use of several heuristics and optimizations [93]; however, a number of these techniques do not work when the underlying logic allows some form of equality as in the case of hybrid logics. When nominals and satisfaction operators are added, the performance of tableau-based theorem provers is affected. This motivated research on possible alternatives, such as the resolution calculus. The best automated theorem provers for first-order logic are based on resolution, and we have already seen many similarities between hybrid and first-order logics.

Resolution calculi for $\mathcal{H}(@,\downarrow)$ and its sublanguages were introduced in [10,11]. In a recent paper [14], the calculus for $\mathcal{H}(@)$ was refined to include ordering and selection functions (see [17] for the definitions of these standard notions). The rules are shown in Figure 10. In the figure, $S(C)$ is a selection function and $\succ$ is an admissible order; furthermore, the main premise of each rules is on the right. The calculus works on formulas in negation normal form (i.e., negation can only appear on atomic formulas), and hence an explicit rule for negation is not required. To extend the calculus to $\mathcal{H}(@,\downarrow)$, simply add the rule

$$(\downarrow) \quad \frac{Cl \cup \{@_t{\downarrow}x.\varphi\}}{Cl \cup \{@_t\varphi[x/t]\}}.$$

Given a formula φ (in negation normal form), let *ClSet*$(\varphi) = \{\{@_i\varphi\}\}$, where i is a nominal not occurring in φ. Define *ClSet*$^*(\varphi)$ — the saturated set of clauses for φ — as the smallest set that includes *ClSet*(φ) and is saturated under the rules of Figure 10 (where saturation means that whenever there are sets matching the antecedent of any rule in *ClSet*$^*(\varphi)$ then also the sets in the consequent should be in *ClSet*$^*(\varphi)$). Then φ is unsatisfiable if and only if $\{\} \in$ *ClSet*$^*(\varphi)$.

The calculus $\mathbf{R}_{\mathcal{H}(@)}$ is implemented in the automated theorem prover HyLoRes [15], which uses an ordering that ensures termination while preserving soundness and completeness.

Example 5.5 We prove that $\downarrow x.\langle R\rangle(x \wedge p) \rightarrow p$ is a tautology in $\mathbf{R}_{\mathcal{H}(@,\downarrow)}$. Consider the clause set corresponding to the negation of the formula:

$$(\wedge)\ \frac{Cl \cup \{@_t(\varphi_1 \wedge \varphi_2)\}}{\begin{array}{c} Cl \cup \{@_t\varphi_1\} \\ Cl \cup \{@_t\varphi_2\}\end{array}} \qquad (\vee)\ \frac{Cl \cup \{@_t(\varphi_1 \vee \varphi_2)\}}{Cl \cup \{@_t\varphi_1, @_t\varphi_2\}}$$

$$(\text{RES})\ \frac{Cl_1 \cup \{@_t\varphi\} \quad Cl_2 \cup \{@_t\neg\varphi\}}{Cl_1 \cup Cl_2}$$

$$([r])\ \frac{Cl_1 \cup \{@_t\langle r\rangle s\} \quad Cl_2 \cup \{@_t[r]\varphi\}}{Cl_1 \cup Cl_2 \cup \{@_s\varphi\}} \qquad (\langle r\rangle)\ \frac{Cl \cup \{@_t\langle r\rangle\varphi\}}{\begin{array}{c} Cl \cup \{@_t\langle r\rangle n\} \\ Cl \cup \{@_n\varphi\}\end{array}} \ \begin{array}{l}\text{for } n \text{ a new nominal} \\ \text{and } \varphi \notin \mathsf{NOM}\end{array}$$

$$(@)\ \frac{Cl \cup \{@_t@_s\varphi\}}{Cl \cup \{@_s\varphi\}} \qquad (\text{REF})\ \frac{Cl \cup \{@_t\neg t\}}{Cl}$$

$$(\text{SYM})\ \frac{Cl \cup \{@_s t\}}{Cl \cup \{@_t s\}}\ \text{if } t \succ s \qquad (\text{PARAM})\ \frac{Cl_1 \cup \{@_s t\} \quad Cl_2 \cup \{\varphi(s)\}}{Cl_1 \cup Cl_2 \cup \{\varphi(s/t)\}}\ \begin{array}{l}\text{if } s \succ t \text{ and} \\ \varphi(s) \succ @_s t\end{array}$$

Restrictions: Let φ and ψ be the displayed formulas in each of the above rules:

- Let $C = C' \cup \{\varphi\}$ be the main premise, then either $S(C) = \{\varphi\}$ or, otherwise, $S(C) = \emptyset$ and $\{\varphi\} \succ C'$.
- Let $D = D' \cup \{\psi\}$ be the auxiliary premise, then $\{\psi\} \succ D'$ and $S(D) = \emptyset$.

Fig. 10. Rules for the resolution calculus $\mathbf{R}_{\mathcal{H}(@)}$ with Order and Selection Functions

1. $\{@_i((\downarrow x.\neg[R]\neg(x\underline{\wedge}p)) \wedge \neg p)\}$ by $(\wedge)$
1. $\{@_i\underline{\downarrow x}.\neg[R]\neg(x \wedge p)\}$, $\{@_i\neg p\}$ by $(\downarrow)$
2. $\{@_i\underline{\neg[R]}\neg(i \wedge p)\}$, $\{@_i\neg p\}$ by $(\langle r\rangle)$
3. $\{@_i\neg[R]\neg j\}$, $\{@_j(i\underline{\wedge}p)\}$, $\{@_i\neg p\}$ by $(\wedge)$
4. $\{@_j\underline{i}\}$, $\{@_{\underline{j}}p\}$, $\{@_i\neg p\}$ by (PARAM)
5. $\{@_i\underline{p}\}$, $\{@_i\underline{\neg p}\}$ by (RES)
6. $\{\}$.

6 Relation with Other Fields

In various areas, hybrid logics has been proposed as a convenient extension of modal logics, either because they give rise to smoother proof systems, or because of their greater expressive power. In this section we briefly discuss a number of cases, and provide pointers to the literature.

Temporal Logic. As indicated in the work of Prior and Bull, hybrid languages allow us to make explicit references to specific times (days, dates, years, etc.), and also to cope with temporal indexicals (such as yesterday, today, tomorrow and now). In addition, many temporally relevant frame properties (such as irreflexivity, asymmetry and trichotomy) that cannot be defined by means of modal formulas can be defined with nominals [28]. When nominals and satisfaction operators are added to an interval-based logic, the result is a $\mathsf{Holds}(t, \varphi)$-driven interval logic similar to those introduced in AI by James Allen [2] (where the satisfaction operators play the role of Holds). By making explicit temporal references possible (combining nominals, satisfaction operators and temporal modalities, one can directly express temporal relations between instants or intervals), hybrid logics remove a serious obstacle to a modal analysis of temporal representation and reasoning.

Nominal tense logics have been studied in detail in [21]. The complexity of the satisfiability problem for a number of hybrid temporal logics is investigated in [7,66]. The minimal hybrid tense logic $\mathcal{H}(\langle R^{-1}\rangle)$

is EXPTIME over the class of all frames and the class of transitive frames, but the complexity drops to NP-complete over the usual frames for linear time (strict total orders), and to PSPACE-complete over the usual frames for branching time (transitive trees). In [7,110,66], results are also given for hybrid languages with the Since and Until operators. Hybrid interval logics were recently studied in [95].

Indexicality and Direct Reference. Hybrid languages are also a powerful resource for studying indexicality in natural language, as an alternative to the more classical use of multi-dimensional modal logic. In the multi-dimensional modal approach, formulas are evaluated at sequences of points, where one point of the sequence is thought of as the point of evaluation, while the others are used as memory locations to store references [96,146,67,52,53]. Hybrid languages move multi-dimensional logic's sequence of evaluation points from the meta-language to the object language, with hybrid variables acting as names for indices (see [24]), and allowing in this way a natural treatment of such indexicals as 'today.' Moreover, when equipped with the @ operator, hybrid languages offer the 'de-scoping' behavior typical of such multi-dimensional operators as here and there. There are also links between hybrid logic and mathematical aspects of multi-dimensional modal logic, particularly the multi-dimensional modal perspective on cylindric algebra (cf. [106]), as $\downarrow$ and @ can be considered as explicit substitution devices.

Feature Logic. Most unification-based approaches to natural language grammar, such as PATR-II, use attribute value matrices (AVMs) to represent feature structures, where re-entrance in the feature structures is represented by "tags" in the AVMs [123]. There is a tight connection between AVMs and deterministic multi-modal logic, except that there is no clear way to express re-entrance in modal logic. As it turns out, the tags that are used to enforce re-entrance in AVMs correspond in a very natural way to nominals in hybrid logic. Thus, adding nominals is enough to make re-entrance expressible.

Previous approaches to encoding re-entrance in modal logic used more complicated techniques. In particular, Kasper-Rounds logic is essentially a fragment of deterministic propositional dynamic logic with program intersection, where the intersection is used to encoding re-entrance. See [33,23,120] for further details.

Dynamic Logic. As we discussed in Section 2.2, hybrid languages were rediscovered, many years after the work of Prior and Bull, by a group of logicians at the Sofia University in Bulgaria. Gargov, Passy and Tinchev were interested in neat axiomatizations of operators in PDL, and they realized that certain operators (e.g., union of programs) are easily captured, whereas others (e.g., program intersection or complement) require extra expressive power. In [113] it is shown that adding nominals is enough to enable natural and succinct characterization of these operators. Adding other kinds of "constants" to the language permits the representation of notions like determinism and looping [74]. In addition, the work of the Sofia school shows how nominals can be used to simplify the construction of models during completeness proofs [114]. See [115] for an excellent overview on combinatory dynamic logics.

For a modern discussion of PDL with nominals (in the framework of description logics) and some new complexity results see [56,55].

Description Logics. Descriptions logics (DLs) are a family of formalisms that allow the representation of, and reasoning about, conceptual knowledge, in a structured and semantically well-understood manner [16]. They evolved from the original knowledge maintenance system KL-ONE of Brachman and Schmolze [41]. Description logics are discussed in detail in Chapter **??** of this handbook.

In [125] Schild identifies a close connections between description logics and modal logics, and uses it to transfer complexity and axiomatization results between the two areas. This connection is established at the level of concepts: *concepts* in description logic are shown to correspond to *formulas* in modal logic. Description logics, however, usually have two levels of representation. The first level is that of *concepts*, which, like modal formulas, denote subsets of the domain. The second level is that of *terminology boxes* (TBoxes) and *assertion boxes* (ABoxes). Using these, one can specify global conditions on models, such as the 'concept inclusion' $C \sqsubseteq D$, which requires that every individual satisfying the concept C should also satisfy the concept D, and the 'assertion' $a{:}C$, which requires that the individual a satisfies the concept C. The basic modal language is not rich enough to express such constructions. By lifting the correspondence to

Converse PDL, Schild managed to account for inference with TBoxes. De Giacomo and Lenzerini [56,55] further extended these results by encoding also ABoxes in Converse PDL.

While the embedding of DLs into Converse PDL have proved useful, it has two important disadvantages. Complexity-wise, the satisfiability problem of Converse PDL is already EXPTIME-complete and, hence, optimal complexity results cannot always be obtained with this technique. Moreover, the model theory of Converse PDL is complicated, due to the presence of the Kleene star (which requires a weak form of induction). Using the extended expressive power of hybrid languages, assertions can be encoded using satisfaction operators, and concept inclusions can be expressed using the universal modality A. See [12,4,36] for detailed discussions on the connections between hybrid and description logics.

Nominals have in fact been independently introduced in DLs. Very early systems like CLASSIC [40] and LOOM [104] already included a form of nominals in the late 80s. Such systems allowed a concept constructor called $\mathcal{O}$ (for "one-of") which permitted enumeration of individuals in the domain of a model. One-of expressions are in fact the same as disjunctions of nominals. The interest in the $\mathcal{O}$ operator dropped during the following years because of complexity issues (as we have seen in Section 4, the presence of nominals can lead to an increase in complexity, and even to undecidability, in the presence of other operators). However, the topic has recently regained interest, as direct reference to individuals seems to be a must for languages for the semantic web, one of the most important modern applications of DLs [94,91]. The $\mathcal{O}$ operator is now part of the W^{3}C-recommended web ontology language OWL [107].

Information Systems. Nominals have turned up in yet another setting, namely the Polish tradition of modal logics for information systems initiated by Pawlak (see [112]). Themes in this tradition include the development of modal logics of similarity (or relative similarity) and there are strong links with the tradition of rough-set theory. Konikowska [97] has proposed adding nominals to such logics. Her work is motivated primarily by proof-theoretical considerations: the ability to name states leads to simpler and more intuitive proof systems.

Logics of Space. Nominals have found several applications in modal logics of space. In this chapter, we have treated hybrid languages from a relational perspective, viewing them as language for describing relational structures. Another well known semantics for modal logics is in terms of topological spaces [108]. A topological space is a tuple $\langle X, \Omega\rangle$ where X is a nonempty set and Ω is a collection of subsets of X satisfying three conditions: X and $\emptyset$ are elements of Ω, every union of elements of Ω is in Ω, and every intersection of finitely many elements of Ω is in Ω. A topological model for the basic modal language, now, consists of a topological space $\langle X, \Omega\rangle$ and a valuation $V : \mathsf{PROP} \rightarrow \wp(X)$. The truth definition for modal formulas with respect to such topological models is similar to the one for Kripke models, except that the modal operator $\Box$ is interpreted as follows (where $m \in X$): $\langle X, \Omega, V\rangle, m \models \Box\varphi$ iff $\exists O \in \Omega$ such that $m \in O$ and for all $m' \in O, \langle X, \Omega, V\rangle, m' \models \varphi$. This topological semantics is useful for spatial reasoning [19,1] and modelling knowledge [54]. As in the relational semantics, we can study notions such as validity of a modal formula on a topological space, and modally definable properties of topological spaces. It turns out that, as a language for defining properties of topological spaces, the basic modal language is very weak. In particular, none of the familiar topological separation axioms is modally definable [71].

Nominals can be introduced in topological models in the same way as in Kripke models: they are simply propositional variables whose valuation is always a singleton set. It was noted in [71] that, with the help of nominals, more properties of topological spaces can be defined, including the separation axioms T_0 and T_1. Sustretov [134] has recently proved a topological analogue of Theorem 3.19, characterizing the properties of topological spaces that can be defined by means of $\mathcal{H}(@)$- and $\mathcal{H}(\mathsf{E})$-formulas. Heinemann [88,87] has investigated hybrid extensions of the bi-modal logic of knowledge and effort presented in [54], in order to obtain complete axiomatization of frame classes that, while relevant for applications, are not expressible in the basic modal language. In [88], Heinemann provides an axiomatization of the class of linear set spaces, using nominals that denote pairs in $X \times \Omega$. In [87], instead, two sorts of nominals are introduced, ranging over elements of X and Ω, respectively, and topological notions like separation and connectedness are axiomatized.

Nominals have also found applications in logics of metric spaces [101].

Second Order Propositional Modal Logic. In [62], the extension of the basic modal language with propositional quantifiers $\exists p$ and $\forall p$ is studied. This language is called *second order propositional modal logic* (SOPML). It was shown in [136,137] that there is a close connection between SOPML and $\mathcal{H}(@, \downarrow)$:

Theorem 6.1 *Every nominal free $\mathcal{H}(@, \downarrow)$-sentence is equivalent to a formula of SOPML. Conversely, if a formula of SOPML has a first-order equivalent, then it is equivalent to a nominal free $\mathcal{H}(@, \downarrow)$-sentence.*

Theorem 6.1 shows that, in some sense, nominal-free $\mathcal{H}(@, \downarrow)$ is the intersection of SOPML and first-order logic. This connection was used in [136,137] to transfer a number of expressivity and frame definability results from $\mathcal{H}(@, \downarrow)$ to SOPML. For example, a first-order formula with one free variable is equivalent to a SOPML-formula iff it is invariant under generated submodels; and an elementary class of frames is definable in SOPML iff it is closed under generated submodels and reflects point-generated subframes (see Theorem 3.24). For more information about SOPML, see Chapter **??** of this handbook.

Modal Predicate Logics. Nominals can also be added on top of a first-order modal basis (cf. Chapter **??** of this handbook). Blackburn and Marx [29] investigate tableau systems for such first-order hybrid logics, while Braüner [44] discusses natural deduction systems. As in the propositional case, the outcome seems to be a better behaved logical system, that comes with general completeness results.

First-order hybrid logics also have advantages in relation to interpolation and Beth definability. Fine [63] showed that interpolation and the Beth definability property fail for quantified S5 with varying domains, and also for any quantified modal logic between K and S5 with constant domains. In [9] it is shown that these properties are regained when state variables, satisfaction operators and $\downarrow$ are added to the language. Actually, interpolation and the Beth property hold relative to any bounded fragment definable class of skeletons (the first-order modal analogue of frames), with either varying, expanding, contracting or constant domains. Moreover, the interpolant can be obtained constructively using the techniques of [30].

For further details on first-order hybrid logics, see Chapter **??** in this handbook.

Labeled Deduction. In [69] the notation $l{:}\varphi$ is introduced, where the meta-linguistic symbol : associates the meta-linguistic label l with the object language formula φ. Labeled deduction proceeds by manipulating such expressions, using the labels to guide proof search. Labelled deduction has been successfully used to provide complete and well behaved calculi for a wide range of logics, including non-classical logics where the notion of "state" is usually crucial (see, e.g., [145]). For example, Simpson defines in [132] a family of labeled natural deduction calculi for modal intuitionistic logics and shows that they have good proof theoretic properties; while Kurtonina [99] uses labels to provide complete calculi for categorial type logics, for a variety of frame classes.

One way to see why hybrid languages are proof-theoretically natural, is to observe that nominals and satisfaction operators can capture the main ideas of labeled deduction. Hybrid languages "internalize" labeled deduction into the object language: nominals are essentially object-level labels, and the formula $@_l\varphi$ asserts in the object language what $l{:}\varphi$ asserts in the meta-language. Internalization in the particular case of tableaux is discussed in [25], while the case of sequent calculus is presented in [130]. We have seen examples of such calculi in Section 5. In a recent paper, Braüner and de Paiva discuss similar internalized calculi for hybrid versions of intuitionistic modal logics [45].

Model Checking. In this chapter we take satisfiability and consequence as the main inference problems, but other reasoning tasks are also relevant for many applications.

In [65] Franceschet and de Rijke investigate the following model checking problem for a number of hybrid logics: given a model, or a model and an assignment in case of languages with binders, and a formula φ decide whether there is a state in the model satisfying φ. They provide algorithms for model checking and investigate their complexity. Their main results are summarized in Figure 11, where k is the length of the input formula, n and m are the number of nodes and edges in the model, respectively, and r is the nesting degree of hybrid binders. Names listed as $\mathcal{DH}(\cdot)$ correspond to hybrid extensions of converse propositional dynamic logic. We can see that the presence of binders makes model checking PSPACE-complete (as complex as model checking

$\mathcal{H}(\langle R^{-1}\rangle, \mathsf{E}, @)$	$\mathrm{O}(k \cdot (n+m))$
$\mathcal{H}(U, S, \mathsf{E}, @)$	$\mathrm{O}(k \cdot n \cdot m)$
$\mathcal{DH}(\mathsf{E}, @)$	$\mathrm{O}(k \cdot (n+m))$
$\mathcal{H}(\downarrow)$	PSPACE-complete
$\mathcal{H}(\mathsf{E}, @, \downarrow)$	PSPACE-complete
$\mathcal{H}_r(\mathsf{E}, @, \downarrow)$	$\mathrm{O}(k \cdot (n+m) \cdot n^r)$
$\mathcal{DH}_r(\mathsf{E}, @, \downarrow)$	$\mathrm{O}(k \cdot (n+m) \cdot n^r)$
$\mathcal{H}(\exists)$	PSPACE-complete
$\mathcal{H}(\mathsf{E}, @, \exists)$	PSPACE-complete
$\mathcal{H}_r(\mathsf{E}, @, \exists)$	$\mathrm{O}(k \cdot (n+m) \cdot n^{2r})$
$\mathcal{DH}_r(\mathsf{E}, @, \exists)$	$\mathrm{O}(k \cdot (n+m) \cdot n^{2r})$

Fig. 11. Complexity of model checking different hybrid languages

full first-order logic), and it is, in general, exponential in the nesting level of binders. The paper discusses the impact of these results in applications like querying and constraint evaluation over semistructured data.

In [66], a different kind of model checking is investigated, which is used in formal verification. There, a Kripke structure typically represents a computational system, and paths through the structure denote different possible computations. In *linear time model checking*, formulas are evaluated not on the Kripke structure itself, but on the set of paths through it. Actually, two versions of the linear model checking problem can be defined: the existential linear time model checking problem is to determine whether a given formula is satisfied in some path of the model, while the universal linear time model checking problem asks whether the formula is satisfied in all paths.

Unraveling a Kripke structure into a tree carries some complications in the presence of nominals: if the original structure makes a nominal i true in a state which is involved in a cycle (i.e., the state is reachable from itself), the "nominal" i will be true in more than one state after unraveling (actually, the denotation of i will be an infinite set). The authors chose to allow such behavior: the only restriction they make is that nominals denote a single state in the original structure, no other conditions are imposed. Under this interpretation, the complexity of linear time model checking for temporal languages coincides with their hybrid extensions: NP-complete (CONP-complete) for $\mathcal{H}(\langle R^{-1}\rangle, @)$ for existential (universal) linear time model checking, and PSPACE for $\mathcal{H}(U, S, @)$.

The Bounded Fragment. We mentioned the bounded fragment of first-order logic in Section 3.2.2 and in Theorem 3.13 we established its tight connection with $\mathcal{H}(@, \downarrow)$.

Bounded formulas have been considered in the literature for a long time. In set theory, where bounded quantifiers are of the form $\exists x.(x \in y \wedge \varphi)$ and $\forall x.(x \in y \rightarrow \varphi)$, the bounded fragment was introduced in 1965 by Levy [102], under the name Δ_0. Δ_0-formulas of set theory have the desirable property of being set-theoretically absolute, meaning that whether a Δ_0-formula $\varphi(x_1, \ldots, x_n)$ holds of sets $a_1, \ldots, a_n$ is independent of the universe of set theory in which $a_1, \ldots, a_n$ reside (cf., for instance, [18]). Bounded formulas have also been considered in the context of arithmetic, where bounded quantifiers are of the form $\exists x.(x \leq y \wedge \varphi)$ and $\forall x.(x \leq y \rightarrow \varphi)$. In fact, there is a field of research of its own called *bounded arithmetic*, which is connected to complexity theory (in particular, to the polynomial hierarchy), propositional proof theory, and the length of propositional proofs [47].

Around 1966, Feferman and Kreisel [61,60] characterized the bounded fragment as the generated submodel invariant fragment of first-order logic. More precisely, they showed that a first-order formula is equivalent to a bounded formula iff it is invariant under generated submodels. Moreover, it was shown in [60] by means of a cut-free sequent calculus that the bounded fragment has interpolation.

7 Discussion

Hybrid logics form a family of natural extensions of modal logic. The naturalness is confirmed by the fact that nominals have been re-invented on several occasions. Hybrid logics offer two important advantages over modal logics: increased expressive power (e.g., in temporal logic, irreflexivity becomes definable when nominals are added to the language) and a simpler proof theory (there are many proof systems for hybrid logic, and they come with powerful general completeness results). The general question, then, is:

> How much do we gain by extending our language (e.g., how much extra expressive power), and what price do we pay (e.g., what are the complexity theoretic consequences)?

For a number of hybrid languages, we have explored these question systematically in this chapter. In particular, the expressivity of various hybrid languages has been characterized by means of analogues of the Van Benthem theorem and the Goldblatt-Thomason theorem. Concerning complexity, we saw that nominals and satisfaction operators often do not increase the complexity, although in exceptional cases, adding a single nominal can already cause undecidability. For languages containing the $\downarrow$-binder, on the other hand, undecidability seems to be the rule, rather than the exception.

We would like to close this chapter by observing that "hybridization", as an operation on logical languages, can be applied in many contexts. As we discussed, nominals have a natural interpretation not only in the relational (Kripke) semantics, but also in topological and algebraic semantics. The hybrid machinery can be added to the basic modal language, or on top of first-order or higher-order modal languages, and with either a classical or an intuitionistic base. Some of these combinations have been investigated (for instance, several recent papers study topological modal language containing nominals), other remain to be explored.

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