Hybrid Logics:
Characterization, Interpolation and Complexity

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Abstract

Hybrid languages are expansions of propositional modal languages which can refer to (or even quantify over) worlds. The use of strong hybrid languages dates back to at least [Pri67], but recent work (for example [BS98, BT98a, BT99]) has focussed on a more constrained system called $\mathcal{H}(\downarrow, @)$. We show in detail that $\mathcal{H}(\downarrow, @)$ is modally natural. We begin by studying its expressivity, and provide model theoretic characterizations (via a restricted notion of Ehrenfeucht-Fraïssé game, and an enriched notion of bisimulation) and a syntactic characterization (in terms of bounded formulas). The key result to emerge is that $\mathcal{H}(\downarrow, @)$ corresponds to the fragment of first-order logic which is invariant for generated submodels. We then show that $\mathcal{H}(\downarrow, @)$ enjoys (strong) interpolation, provide counterexamples for its finite variable fragments, and show that weak interpolation holds for the sublanguage $\mathcal{H}(@)$. Finally, we provide complexity results for $\mathcal{H}(@)$ and other fragments and variants, and sharpen known undecidability results for $\mathcal{H}(\downarrow, @)$.

1 Introduction

In their simplest form, hybrid languages are modal languages which use *formulas* to refer to worlds. To build a simple hybrid language, take an ordinary language of propositional modal logic (built over some collection of propositional variables p, q, r, and so on) and add a second type of atomic formula. These new atoms are called *nominals*, and are typically written i, j and k. Both types of atom can be freely combined to form more complex formulas in the usual way; for example,

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)$$

is a well formed formula. And now for the key idea: insist that *each nominal must be true at exactly one world in any model*. Thus a nominal names a world by being true there and nowhere else. This simple idea gives rise to richer logics (note, for example, that the previous formula is valid: if the antecedent is satisfied at a world m, then the unique world named by i must be accessible from m, and both p and q must be true there) and enables us to define classes of frames that ordinary modal languages cannot (we'll see some examples later).

Once the idea of using "formulas as terms" has been noted (Arthur Prior [Pri67], influenced by unpublished work of C. A. Meredith, seems to have been the first to grasp its potential) the way lies open for further enrichments. The most obvious is to regard nominals not as names but as *variables over individual worlds*, and to add quantifiers. That is, we now allow expressions like

$$\forall x.\Diamond(x \wedge \exists y.\Diamond(y \wedge \Diamond y))$$

to be well formed. This sentence is satisfied at a world m if and only if from every world x that is accessible from m we can reach at least one reflexive world y. No formula with this property exists in ordinary modal languages, or even in modal languages enriched with nominals. Unsurprisingly, if we are allowed to quantify over worlds in this manner, it is straightforward to define hybrid languages that offer first-order expressivity over models. Early work on hybrid languages (notably Bull [Bul70] and Passy and Tinchev [PT85, PT91]) was largely concerned with such systems.

The idea of binding variables to worlds underlies much current work on hybrid languages, but for many purposes the $\forall$ binder is arguably too strong: $\forall$ obscures the *locality* intuition central to Kripke semantics. Fundamental to Kripke semantics is the relativization of semantic evaluation to worlds. That is, to evaluate a modal formula we need to specify some world m (the *current* world) and begin evaluation *there*. The function of the modalities is to scan the worlds accessible from m, the worlds accessible from those worlds, and so on; in short, m is the starting point for step-wise local exploration of the model. Languages which allow variables to be bound to *arbitrary* worlds don't mesh well with this intuition.

Thus recent work on hybrid languages has focussed on a language called $\mathcal{H}(\downarrow, @)$. This extends the simplest type of hybrid language (propositional modal logic plus nominals) with two new mechanisms, $\downarrow$ and $@$. Now, $\downarrow$ binds variables to worlds, but (unlike $\forall$) it does so in an intrinsically *local* way:

> *The* $\downarrow$ *binder binds variables to the* current *world. In essence it enables us to create a name for the here-and-now*.

The @ operator (which does *not* bind variables) is a natural counterpart to $\downarrow$. Whereas $\downarrow$ "stores" the current world (by binding a variable to it), @ enables us to "retrieve" worlds. More precisely, a formula of the form $@_x\varphi$ is an instruction to move to the world labeled by the variable x and evaluate φ there. Previous work on $\mathcal{H}(\downarrow, @)$ has concentrated on relating it to other hybrid languages [BS98], studying it axiomatically [BT98a], and developing analytic proof techniques [Bla00, Tza99]. Taken together, this work suggests that $\mathcal{H}(\downarrow, @)$ and certain of its sublanguages (notably $\mathcal{H}(@)$) are important systems. The purpose of the present paper is to demonstrate in detail that this impression is justified.

We do so as follows. After defining $\mathcal{H}(\downarrow, @)$ and noting some basic results in Section 2, we turn in Section 3 to the task of characterizing its expressivity. The key result to emerge is this: $\mathcal{H}(\downarrow, @)$ is not merely local, it is the language which *characterizes* locality. More precisely, $\mathcal{H}(\downarrow, @)$ corresponds to the fragment of first-order logic which is invariant under generated submodels. Previous discussions of $\mathcal{H}(\downarrow, @)$ have stressed that it is "modally natural"; our characterization confirms this impression and makes it precise. In Section 3.4 we discuss the consequences of this characterization for frame-definability, completeness, and tense logic. In Section 4, we show that $\mathcal{H}(\downarrow, @)$ is well-behaved in yet another way: it has the strong (arrow) interpolation property, and the sublanguage $\mathcal{H}(@)$ has weak interpolation.

In Section 5 we turn to complexity and decidability. It is known that $\mathcal{H}(\downarrow, @)$ has an undecidable satisfiability problem (indeed, this is clear from the characterization result); but it is also known that $\mathcal{H}(@)$ is decidable. We provide complexity results for $\mathcal{H}(@)$ and other fragments and variants, and sharpen known undecidability results for $\mathcal{H}(\downarrow, @)$. In particular we show that the satisfiability problem for $\mathcal{H}(\downarrow, @)$ is undecidable, even for sentences not containing @, nominals, or propositional variables. Our complexity and undecidability proofs make heavy use of *spypoint arguments*. We close the paper with a discussion of a key open problem.

The paper is largely self contained, but as the literature on hybrid languages is relatively small, it is possible to give the reader a swift overview of what is available. First, two early papers on $\forall$-based hybrid languages (namely [Bul70] and [PT91]) deserve to be more widely read: both contain important technical ideas and interesting motivation for the use of hybrid languages. Second, some work has been done on very basic hybrid languages (that is, modal or tense languages enriched with nominals, but with no additional mechanisms such as $\downarrow$, @, or $\forall$); early references here are [GG93] and [Bla93]. Third, while [BT98a] and [BT99] are the basic references for $\mathcal{H}(\downarrow, @)$, an interesting discussion of $\downarrow$ as part of a stronger system can be found in [Gor96]. Finally, in addition to the proof theoretical investigations of [Bla00] and [Tza99], there is [Sel91, Sel97]. For a more detailed guide to the field, see the hybrid logic home-page (`http://turing.wins.uva.nl/~carlos/hybrid`).

2 Preliminaries

In this section we define the syntax and semantics of $\mathcal{H}(\downarrow, @)$ and note some of its basic properties.

Definition 2.1 (Language) *Let* $\mathsf{PROP} = \{p_1, p_2, \ldots\}$ *be a countable set of* propositional variables*,* $\mathsf{NOM} = \{i_1, i_2, \ldots\}$ *a countable set of* nominals*, and* $\mathsf{WVAR} = \{x_1, x_2, \ldots\}$ *a countable set of* world variables*. We assume that* PROP*,* NOM *and* WVAR *are pairwise disjoint. We call* $\mathsf{WSYM} = \mathsf{NOM} \cup \mathsf{WVAR}$ *the set of* world symbols*, and* $\mathsf{ATOM} = \mathsf{PROP} \cup \mathsf{NOM} \cup \mathsf{WVAR}$ *the set of* atoms*. The* well-formed formulas *of the hybrid language (over the signature* $\langle \mathsf{PROP}, \mathsf{NOM}, \mathsf{WVAR} \rangle$*) are*

$$\varphi := \top \mid a \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \Box\varphi \mid \downarrow x_j.\varphi \mid @_s\varphi$$

where $a \in \mathsf{ATOM}$*,* $x_j \in \mathsf{WVAR}$ *and* $s \in \mathsf{WSYM}$*. Let* $\mathcal{L}$ *be the set of all well-formed formulas. For* $T \subseteq \mathcal{L}$*,* $\mathsf{PROP}(T)$*,* $\mathsf{NOM}(T)$ *and* $\mathsf{WVAR}(T)$ *denote, respectively, the set of propositional variables, nominals, and world variables which occur in formulas in* T *(we drop brackets in the usual way when* T *is a singleton set).* $I\!P(T)$ *will denote* $\mathsf{PROP}(T) \cup \mathsf{NOM}(T)$*, and will be called the* language *of* T*.*

In what follows we assume that a signature $\langle \mathsf{PROP}, \mathsf{NOM}, \mathsf{WVAR} \rangle$, and hence $\mathcal{L}$, has been fixed. We usually write p, q and r for propositional variables, i, j and k for nominals, and x, y and z for world variables. As usual, $\Diamond\varphi$ is defined to be $\neg\Box\neg\varphi$

Note that all three types of atomic symbol are *formulas*. Further, note that the above syntax is simply that of ordinary unimodal propositional modal logic extended by the clauses for $\downarrow x_j.\varphi$ and $@_s\varphi$. Finally, the difference between nominals and world variables is simply this: nominals cannot be bound by $\downarrow$, whereas world variables can. In fact, nominals could be dispensed with (it is always possible to make do with free world variables instead) but sometimes it is useful to have a special kind of world symbol that can't be accidentally bound.

Definition 2.2 *The notions of* free *and* bound *world variable,* substitution*, and of a world symbol* t being substitutable for x in φ*, are defined in the manner familiar from first-order logic, with* $\downarrow$ *as the only binding operator. We use* $\varphi[t/s]$ *to denote the formula obtained by replacing all free instances of the world symbol* t *by the world symbol* s*.*

A sentence *is a formula containing no free world variables. A formula is* pure *if it contains no propositional variables, and* nominal-free *if it contains no nominals.*

Definition 2.3 (Semantics) *A (hybrid)* model $\mathfrak{M}$ *for* $\mathcal{L}$ *is a triple* $\mathfrak{M} = \langle M, R, V \rangle$ *such that* M *is a non-empty set,* R *a binary relation on* M*, and* $V : \mathsf{PROP} \cup \mathsf{NOM} \longrightarrow Pow(M)$ *such that for all nominals* $i \in \mathsf{NOM}$*,* $V(i)$ *is a singleton subset of* M*. (We use gothic letters* $\mathfrak{M}$ *for models, italic roman* M *for their domains.) We usually call the elements of* M worlds *(though sometimes we call* *them times or states),* R *the* accessibility relation*, and* V *the* valuation*. A* frame *is a pair* $\mathfrak{F} = \langle M, R\rangle$*; that is, a frame is a model without a valuation.*

An assignment g *for* $\mathfrak{M}$ *is a mapping* $g : \mathsf{WVAR} \longrightarrow M$*. Given an assignment* g*, we define* g^x_m *(an* x-variant of g*) by* $g^x_m(x) = m$ *and* $g^x_m(y) = g(y)$ *for* $x \neq y$*.*

Let $\mathfrak{M} = \langle M, R, V\rangle$ *be a model,* $m \in M$*, and* g *an assignment. For any atom* a*, let* $[V, g](a) = \{g(a)\}$ *if* a *is a world variable, and* $V(a)$ *otherwise. Then the* forcing relation *is defined as follows:*

$$
\begin{array}{lll}
\mathfrak{M}, g, m \Vdash \top & & \\
\mathfrak{M}, g, m \Vdash a & \textit{iff} & m \in [V, g](a),\ a \in \mathsf{ATOM} \\
\mathfrak{M}, g, m \Vdash \neg\varphi & \textit{iff} & \mathfrak{M}, g, m \nVdash \varphi \\
\mathfrak{M}, g, m \Vdash \varphi \wedge \psi & \textit{iff} & \mathfrak{M}, g, m \Vdash \varphi \textit{ and } \mathfrak{M}, g, m \Vdash \psi \\
\mathfrak{M}, g, m \Vdash \Box\varphi & \textit{iff} & \forall m'.(Rmm' \Rightarrow \mathfrak{M}, g, m' \Vdash \varphi) \\
\mathfrak{M}, g, m \Vdash \downarrow x.\varphi & \textit{iff} & \mathfrak{M}, g^x_m, m \Vdash \varphi \\
\mathfrak{M}, g, m \Vdash @_s\varphi & \textit{iff} & \mathfrak{M}, g, m' \Vdash \varphi, \textit{ where } [V, g](s) = \{m'\},\ s \in \mathsf{WSYM}.
\end{array}
$$

When $\mathfrak{M}$ *and* g *are understood from context we will simply write* $m \Vdash \varphi$ *for* $\mathfrak{M}, g, m \Vdash \varphi$*. We write* $\mathfrak{M}, g \Vdash \varphi$ *iff for all* $m \in M$*,* $\mathfrak{M}, g, m \Vdash \varphi$*. We write* $\mathfrak{M} \models \varphi$ *iff for all* g*,* $\mathfrak{M}, g \Vdash \varphi$*.*

The first five clauses are essentially the standard Kripke forcing relation for propositional modal logic; the only difference is that whereas the standard definition relativizes semantic evaluation to worlds m, we relativize to variable assignments g as well. Note that the second clause covers all three types of atom (propositional variables, nominals, and world variables) and that given any model $\mathfrak{M}$ and assignment g, any world symbol (whether it is a nominal or a world variable) will be forced at a *unique* world; this is an immediate consequence of the way we defined valuations and assignments. As promised in the introduction, $\downarrow$ binds world variables to the world where evaluation is being performed, and $@_s$ shifts evaluation to the world named by s. Just as in first-order logic, if φ is a *sentence* it is irrelevant which assignment g is used to perform evaluation: $\mathfrak{M}, g, m \Vdash \varphi$ for *some* assignment g iff $\mathfrak{M}, g, m \Vdash \varphi$ for *all* assignments g. Hence for sentences the relativization to assignments of the forcing relation can be dropped, and we simply write $\mathfrak{M}, m \Vdash \varphi$ instead of $\forall g.(\mathfrak{M}, g, m \Vdash \varphi)$.

A formula φ is *satisfiable* if there is a model $\mathfrak{M}$, an assignment g on $\mathfrak{M}$, and a world $m \in M$ such that $\mathfrak{M}, g, m \Vdash \varphi$. A formula φ is *valid* if for all models $\mathfrak{M}$, $\mathfrak{M} \models \varphi$. A formula φ is a *local consequence* of a set of formulas T if for some finite subset $\{\varphi_1, \ldots, \varphi_n\}$ of T, $\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \varphi$ is valid. A formula φ is a *global consequence* of a set of formulas T if for all models $\mathfrak{M}$, $\mathfrak{M} \models \varphi$ only if for all $\psi \in T$, $\mathfrak{M} \models \psi$. We denote local consequence by $T \models \varphi$ and global consequence by $T \models^{glo} \varphi$. As in ordinary propositional modal logic, local consequence is strictly stronger than global consequence.

$\mathcal{H}(\downarrow, @)$ offers us considerable expressive power over models. For example we can define the *Until* operator:

$$\textit{Until}(\varphi, \psi) \ := \downarrow x.\Diamond \downarrow y.@_x(\Diamond(y \wedge \varphi) \wedge \Box(\Diamond y \rightarrow \psi)).$$

Note how this works: we name the current world x, use $\Diamond$ to move to an accessible world, which we name y, and then use @ to jump us back to x. We then use the modalities to insist that (1) φ holds at the world named y, and (2) ψ holds at all successors of the current world that precede this y-labeled world.

But there is an obvious (and modally natural) limit to the expressive power of $\mathcal{H}(\downarrow, @)$: any nominal-free sentence is preserved under the formation of *point-generated* (or *rooted*) submodels. That is, if a sentence φ is satisfied at a world m in a model $\mathfrak{M}$, and we form a submodel $\mathfrak{M}_m$ by discarding from $\mathfrak{M}$ all the worlds that are *not* reachable by making a finite (possibly empty) sequence of transitions from m, then $\mathfrak{M}_m$ also satisfies φ at m. (The key point to observe is that in any subformula of φ of the form $@_t\psi$, t must be a world variable bound by some previous occurrence of $\downarrow$. As $\downarrow$ binds to the current world, t is bound to some world in the submodel generated by m, thus φ is unaffected by the restriction to $\mathfrak{M}_m$.) That is, $\mathcal{H}(\downarrow, @)$ is genuinely local: only reachable worlds are relevant to semantic evaluation. In the following section we shall generalize this observation (we have not merely preservation, but *invariance*) and show that it characterizes the expressivity of $\mathcal{H}(\downarrow, @)$.

$\mathcal{H}(\downarrow, @)$ also offers us considerable expressive power with respect to *frames*. Modal logicians like to view modal languages as tools for talking about frames, and they do so via the concept of frame validity. A formula φ is *valid on a frame* $\mathfrak{F} = \langle M, R\rangle$ if for every valuation V on $\mathfrak{F}$, and every assignment g on $\mathfrak{F}$, and every $m \in M$, $\langle \mathfrak{F}, V\rangle, g, m \Vdash \varphi$. A formula is *valid on a class of frames* F if it is valid on every frame $\mathfrak{F}$ in F. A formula φ *defines a class of frames* if it is valid on precisely the frames in F, and it defines a *property* of frames (for example, transitivity of the accessibility relation) if it defines the class of frames with that property. Many interesting properties are definable using pure, nominal-free, sentences:

$\downarrow x.\Box\neg x$	*Irreflexivity*
$\downarrow x.\Box\Box\neg x$	*Asymmetry*
$\downarrow x.\Box(\Diamond x \rightarrow x)$	*Antisymmetry*
$\downarrow x.\Box \downarrow y.@_x\Diamond\Diamond y$	*Density*
$\downarrow x.\Box\Box \downarrow y.@_x\Diamond y$	*Transitivity*
$\downarrow x.\Diamond \downarrow y.@_x(\Box\Box\neg y \wedge \Box \downarrow z.@_y(z \vee \Diamond z))$	*Right-Discreteness*

With the exception of transitivity and density, none of these properties are definable in ordinary modal logic. In Section 3.4 we shall characterize the classes of frames that pure, nominal-free, sentences can define.

[BT99] provides the following complete axiom system for $\mathcal{H}(\downarrow, @)$:

Definition 2.4 (Axiomatization) *Let φ, ψ be formulas, v a metavariable over world variables and s, t metavariables over world symbols. The hybrid logic $\mathsf{K}[\mathcal{H}(\downarrow, @)]$ is the smallest subset of $\mathcal{L}$ containing all instances of propositional tautologies, all instances of the following axiom schemes, and closed under the following deduction rules:*

MP. $\vdash \varphi \rightarrow \psi, \vdash \varphi \Rightarrow \vdash\psi$.

K. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.
N. $\vdash \varphi \Rightarrow \vdash\Box\varphi$.

Q1. $\downarrow v.(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \downarrow v.\psi)$, φ *without free occurrences of* v.
Q2. $\downarrow v.\varphi \rightarrow (s \rightarrow \varphi[v/s])$, s *substitutable for* v *in* φ.
Q3. $\downarrow v.(v \rightarrow \varphi) \rightarrow \downarrow v.\varphi$.
Self Dual$_\downarrow$. $\downarrow v.\varphi \leftrightarrow \neg \downarrow v.\neg\varphi$.
N$_\downarrow$. $\vdash \varphi \Rightarrow \vdash \downarrow v.\varphi$.

K$_@$. $@_s(\varphi \rightarrow \psi) \rightarrow (@_s\varphi \rightarrow @_s\psi)$.
Self Dual$_@$. $@_s\varphi \leftrightarrow \neg @_s\neg\varphi$.
Introduction. $(s \wedge \varphi) \rightarrow @_s\varphi$.
Label. $@_s s$.
Nom. $@_s t \rightarrow (@_t\varphi \rightarrow @_s\varphi)$.

Swap. $@_s t \leftrightarrow @_t s$.
Scope. $@_t @_s \varphi \leftrightarrow @_s \varphi$.
$N_@$. $\vdash \varphi \Rightarrow \vdash @_s \varphi$.

Back. $\Diamond @_s \varphi \to @_s \varphi$.
Bridge. $(\Diamond s \wedge @_s \varphi) \to \Diamond \varphi$.

Paste-0. $\vdash @_s(t \wedge \varphi) \to \psi \Rightarrow \vdash @_s \varphi \to \psi$, $t \in \mathsf{WSYM} \backslash \mathsf{WSYM}(\{\varphi, \psi, s\})$.
Paste-1. $\vdash @_s \Diamond(t \wedge \varphi) \to \psi \Rightarrow \vdash @_s \Diamond \varphi \to \psi$, $t \in \mathsf{WSYM} \backslash \mathsf{WSYM}(\{\varphi, \psi, s\})$.

We use this axiomatization to help prove strong interpolation in Section 4. For this purpose, the following derived theorems will be useful; their derivation from the above axiomatization is a simple exercise; see [BT99, Lemma 4.1] and [BT98b, Lemma 7] for details.

Proposition 2.5

i. *The formula schemes $K_\downarrow$ and $Distr_@$ belong to $\mathsf{K}[\mathcal{H}(\downarrow, @)]$, where $K_\downarrow$ is $\downarrow v.(\varphi \to \psi) \to (\downarrow v.\varphi \to \downarrow v.\psi)$, and $Distr_@$ is $@_s(\varphi \wedge \psi) \leftrightarrow (@_s \varphi \wedge @_s \psi)$.*
ii. *If $\vdash \varphi$ and i is a nominal in φ, then for some world variable x not occurring in φ, $\vdash \downarrow x.\varphi[i/x]$. If $\vdash \varphi$ and x is a free variable in φ, then for some nominal i not occurring in φ, $\vdash \varphi[x/i]$.*

But we are interested in this axiomatization for another reason. The completeness result proved in [BT99] is very general: not only does this axiomatization generate all valid formulas, but it automatically extends to many stronger logics. In particular, if we add a pure, nominal-free, sentence φ as an additional axiom, the resulting system is strongly complete with respect to the class of frames that φ defines. In Section 3.4 we shall characterize the completeness results covered by such extensions.

Before starting our investigations, one final remark. We have defined $\mathcal{H}(\downarrow, @)$ to be an expansion of unimodal propositional modal logic. But of course, it would have been equally straightforward to add nominals, variables, @ and $\downarrow$ to a multimodal language (that is, a language containing an indexed collection of modalities, each interpreted by a separate relation) or a language of tense logic (that is, propositional modal logic enriched with a modality which scans the converse of the relation R). For the most part we will work with the above version of $\mathcal{H}(\downarrow, @)$ (most results go through for other formulations essentially unchanged) but sometimes it will be interesting to switch to a richer underlying modal language, especially when we discuss computational complexity in Section 5.

3 Characterizing $\mathcal{H}(\downarrow, @)$

In this section we characterize (the first-order fragment corresponding to) $\mathcal{H}(\downarrow, @)$. We begin by providing a *syntactic* characterization. In particular, we shall first extend the *standard translation* ST of modal logic into first-order logic (cf. [vB83]) to $\mathcal{H}(\downarrow, @)$. It will be clear that the range of our translation lies in a certain *bounded fragment*, and we shall define a reverse translation HT which maps the bounded fragment back into the hybrid language. Thus we are free to think either in terms of $\mathcal{H}(\downarrow, @)$ or the corresponding bounded fragment.

But how are these languages characterized *semantically*? It should be clear that $\mathcal{H}(\downarrow, @)$ is a genuine hybrid of modal and first-order ideas (after all, it combines Kripke semantics with the idea of binding variables to worlds) thus there are two obvious ways to proceed. The first is essentially first-order: we could look for a weaker notion of Ehrenfeucht-Fraïssé game. The second is essentially modal: we could try looking for a stronger notion of bisimulation. We shall pursue *both* options. As we shall see, both yield natural notions of equivalence between models, and by relating them (and drawing on our syntactic characterization) we can provide a detailed picture of what $\mathcal{H}(\downarrow, @)$ offers.

3.1 Translations

We focus on two kinds of signature for first-order logic with equality. First we have *modal* signatures (familiar from modal correspondence theory [vB83]) which consist of one binary predicate R, countably many unary predicates, and no constant symbols. It will be convenient to make the set of first-order variables at our disposal explicit in the signature (just as we did when we defined hybrid signatures in Definition 2.1) thus, a modal signature has the form $\langle \{R\} \cup \mathsf{UN\text{–}REL}, \{\}, \mathsf{VAR}\rangle$. A *hybrid* signature is an expansion of the modal signature with countably many constant symbols, thus hybrid signatures have the form $\langle \{R\} \cup \mathsf{UN\text{–}REL}, \mathsf{CONS}, \mathsf{VAR}\rangle$. Note that any hybrid model $\mathfrak{M} = \langle M, R, V\rangle$ can be regarded as a first-order model with domain M, for the accessibility relation R can be used to interpret the binary predicate R, unary predicates can be interpreted by the subsets V assigns to propositional variables, and constants (if any) can be interpreted by the worlds that nominals name. So we let the context determine whether we are thinking of first-order or hybrid models, and continue to use the notation $\mathfrak{M} = \langle M, R, V\rangle$.

We first extend the well-known *standard translation* to $\mathcal{H}(\downarrow, @)$. The translation ST from the hybrid language over $\langle \mathsf{PROP}, \mathsf{NOM}, \mathsf{WVAR}\rangle$ into first-order logic over the signature $\langle \{R\} \cup \{P_j \mid p_j \in \mathsf{PROP}\}, \mathsf{NOM}, \mathsf{WVAR} \cup \{x, y\}\rangle$ is defined by mutual recursion between two functions ST_x and ST_y. Recall that $\varphi[x/y]$ means "replace all free instances of x by y".

$$
\begin{array}{lcl|lcl}
ST_x(p_j) & = & P_j(x), p_j \in \mathsf{PROP}. & ST_y(p_j) & = & P_j(y), p_j \in \mathsf{PROP}. \\
ST_x(i_j) & = & x = i_j, i_j \in \mathsf{NOM}. & ST_y(i_j) & = & y = i_j, i_j \in \mathsf{NOM}. \\
ST_x(x_j) & = & x = x_j, x_j \in \mathsf{WVAR}. & ST_y(x_j) & = & y = x_j, x_j \in \mathsf{WVAR}. \\
ST_x(\neg\varphi) & = & \neg ST_x(\varphi). & ST_y(\neg\varphi) & = & \neg ST_y(\varphi). \\
ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi). & ST_y(\varphi \wedge \psi) & = & ST_y(\varphi) \wedge ST_y(\psi). \\
ST_x(\Diamond\varphi) & = & \exists y.(Rxy \wedge ST_y(\varphi)). & ST_y(\Diamond\varphi) & = & \exists x.(Ryx \wedge ST_x(\varphi)). \\
ST_x(\downarrow x_j.\varphi) & = & (ST_x(\varphi))[x_j/x]. & ST_y(\downarrow x_j.\varphi) & = & (ST_y(\varphi))[x_j/y]. \\
ST_x(@_s\varphi) & = & (ST_x(\varphi))[x/s]. & ST_y(@_s\varphi) & = & (ST_y(\varphi))[y/s].
\end{array}
$$

For m an element in the domain of a given model $\mathfrak{M}$ we will often write $ST_m(\varphi)$ as shorthand for $ST_x(\varphi)[m]$. This translation differs from the one given in [BT98a]; these authors handle $\downarrow$ as follows:

$$ST_x(\downarrow x_j.\varphi) \quad = \quad \exists x_j.(x = x_j \wedge ST_x(\varphi)).$$

The [BT98a] translation makes the quantificational effect of $\downarrow$ clear, but our translation draws attention to another perspective: in adding $\downarrow$ and $@$ we have in effect enriched the modal language with an *explicit substitution operator*. Such operators are used in the study of cylindric algebras, and were added to cylindric modal logic in [Ven94].

The link between $\downarrow$ and explicit substitution can be made even more clear if we expand the first-order language with an explicit substitution operator (like s^i_j in the theory of cylindric algebras) and adjust our definition of ST to take advantage of it. We do this as follows. Add the following clause to the grammar generating the first-order language: if φ is a formula and x, y are variables, then $S^x_y\varphi$ is a formula. Interpret S^x_y as follows:

$$\mathfrak{M} \models S^x_y\varphi[g] \Leftrightarrow \begin{cases} \mathfrak{M} \models \varphi[g] & \text{for } x = y \\ \mathfrak{M} \models \varphi[g^x_{g(y)}] & \text{for } x \neq y. \end{cases}$$

Clearly $S^x_y\varphi$ and $\varphi[x/y]$ are equivalent. This expansion can be axiomatized by adding the following axiom schemata to a complete axiomatization of first-order logic with equality:

$$
\begin{array}{lcl}
S^x_x\varphi & \leftrightarrow & \varphi \\
S^x_y\varphi & \leftrightarrow & \exists x.(x = y \wedge \varphi) \quad \text{for } x \neq y.
\end{array}
$$

And now we can give transparent translations of $\downarrow$ and @:

$$\begin{array}{lcl} ST_x(\downarrow x_j.\varphi) & = & S_x^{x_j} ST_x(\varphi) \\ ST_x(@_s\varphi) & = & S_s^x ST_x(\varphi). \end{array}$$

Note that theorems like $\downarrow v.@_v\varphi \leftrightarrow \downarrow v.\varphi$ follow almost immediately, for $ST_x(\downarrow v.@_v\varphi) = S_x^v S_v^x ST_x(\varphi)$, which is equivalent to $S_x^v ST_x(\varphi)$ because $S_x^v S_v^x \varphi \equiv S_x^v S_x^x \varphi \equiv S_x^v \varphi$. However we shall stick with our original formulation of ST in what follows.

Proposition 3.1 (ST preserves truth) *Let φ be a hybrid formula. Then for all hybrid models $\mathfrak{M}$, $m \in M$, and assignments g, $\mathfrak{M}, g, m \Vdash \varphi$ iff $\mathfrak{M} \models ST_m(\varphi)[g]$.*

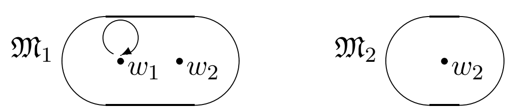
PROOF. A straightforward extension of the induction familiar from ordinary modal logic. The only cases which are new are $ST_x(\downarrow x_j.\varphi)$ and $ST_x(@_s\varphi)$. But $\mathfrak{M}, g, m \Vdash \downarrow x_j.\varphi$, iff $\mathfrak{M}, g_m^{x_j}, m \Vdash \varphi$, by IH iff, $\mathfrak{M} \models ST_m(\varphi)[g_m^{x_j}]$, iff $\mathfrak{M} \models (ST_m(\varphi))[x_j/x][g]$. The argument for $ST_x(@_s\varphi)$ is similar. QED

Now for the interesting question: what is the *range* of ST? In fact it belongs to a *bounded* fragment of first-order logic. Given a first-order signature $\langle \{R\} \cup \mathsf{UN{-}REL}, \mathsf{CONS}, \mathsf{VAR} \rangle$ we define the *bounded* fragment as the set of formulas generated by the following grammar:

$$\varphi := Rtt' \mid P_j t \mid t = t' \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \exists x_i.(Rtx_i \wedge \varphi) \text{ (for } x_i \neq t).$$

where $x_i \in \mathsf{VAR}$ and $t, t' \in \mathsf{VAR} \cup \mathsf{CONS}$.

The side-condition on the generation of existentially quantified formulas is crucial: it prevents sentences like $\exists x.(Rxx \wedge x = x)$ from falling into the fragment. The sentence $\exists x.Rxx$ is probably the simplest example of a first-order sentence which is *not invariant for generated submodels* (or subframes). In fact it is not even *preserved* under the formation of generated submodels, for it is true in the model $\mathfrak{M}_1$ but not in the generated submodel $\mathfrak{M}_2$:



Clearly ST generates formulas in the bounded fragment. Moreover, we can also translate the bounded fragment back into $\mathcal{H}(\downarrow, @)$. The translation HT from the bounded fragment over $\langle \{R\} \cup \mathsf{UN{-}REL}, \mathsf{CONS}, \mathsf{VAR} \rangle$ into the hybrid language over $\langle \mathsf{UN{-}REL}, \mathsf{CONS}, \mathsf{VAR} \rangle$ is defined as follows. For $t, t' \in \mathsf{VAR} \cup \mathsf{CONS}$

$$\begin{array}{lcl} HT(Rtt') & = & @_t \Diamond t'. \\ HT(P_j t) & = & @_t p_j. \\ HT(t = t') & = & @_t t'. \\ HT(\neg\varphi) & = & \neg HT(\varphi). \\ HT(\varphi \wedge \psi) & = & HT(\varphi) \wedge HT(\psi). \\ HT(\exists v.(Rtv \wedge \varphi)) & = & @_t \Diamond \downarrow v.HT(\varphi). \end{array}$$

By construction, $HT(\varphi)$ is a hybrid formula built as a boolean combination of @-formulas (formulas whose main operator is @). We can now prove the following strong truth preservation result.

Proposition 3.2 (HT preserves truth) *Let φ be a bounded formula. Then for every first-order model $\mathfrak{M}$ and for every assignment g, $\mathfrak{M} \models \varphi[g]$ iff $\mathfrak{M}, g \Vdash HT(\varphi)$.*

PROOF. The proof uses the following fact about boolean combinations of @-formulas: for any @-formula φ, there exists an m such that $\mathfrak{M}, g, m \Vdash \varphi$ iff $\mathfrak{M}, g \Vdash \varphi$.

Again there is only one interesting case: $HT(\exists v.(Rtv \wedge \varphi))$. Now, $\mathfrak{M} \models \exists v.(Rtv \wedge \varphi)[g]$ iff $\mathfrak{M} \models (Rtv \wedge \varphi)[g^v_{\mathbf{m}}]$ for some $\mathbf{m} \in M$. Let $\mathbf{t}$ be the interpretation of t in $\mathfrak{M}$ under $g^v_{\mathbf{m}}$. Because of the restriction on variables in bounded quantification, $t \neq v$, whence $\mathbf{t}$ is also the interpretation of t in $\mathfrak{M}$ under g. So $R\mathbf{tm}$ holds in $\mathfrak{M}$ and $\mathfrak{M} \models \varphi[g^v_{\mathbf{m}}]$. By the inductive hypothesis, $\mathfrak{M}, g^v_{\mathbf{m}} \Vdash HT(\varphi)$ iff $\mathfrak{M}, g, \mathbf{m} \Vdash \downarrow v.HT(\varphi)$, iff $\mathfrak{M}, g, \mathbf{t} \Vdash \Diamond \downarrow v.HT(\varphi)$ iff $\mathfrak{M}, g, \mathbf{t} \Vdash @_t \Diamond \downarrow v.HT(\varphi)$ iff $\mathfrak{M}, g \Vdash @_t \Diamond \downarrow v.HT(\varphi)$. QED

As simple corollaries we have:

Corollary 3.3 *Let $\varphi(x)$ be a bounded formula with only x free. Then for all models $\mathfrak{M}$ and for all $m \in M$, $\mathfrak{M} \models \varphi[m]$ iff $\mathfrak{M}, m \Vdash \downarrow x.HT(\varphi)$.*

Corollary 3.4 *Let φ be a first-order formula in the hybrid signature. Then the following are equivalent*

i. φ is equivalent to the standard translation of a hybrid formula.

ii. φ is equivalent to a formula in the bounded fragment.

Moreover, there are effective translations between $\mathcal{H}(\downarrow, @)$ and the bounded fragment.

3.2 Generated back-and-forth systems

We now turn to the problem of providing semantic characterizations of $\mathcal{H}(\downarrow, @)$ (or equivalently, of the bounded fragment). In this section we adopt an essentially first-order approach: we define *generated back-and-forth systems*, basically a restricted form of Ehrenfeucht-Fraïssé game, and link it to the concept of *generated submodels*.

Definition 3.5 (Partial Isomorphism) *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. A function h from a subset of M to a subset of N is called a* partial isomorphism *if*

i. h is a bijection;

ii. for all $x \in dom(h)$, for all $a \in \mathsf{PROP} \cup \mathsf{NOM}$, $x \in V^{\mathfrak{M}}(a)$ iff $h(x) \in V^{\mathfrak{N}}(a)$;

iii. for all $x, y \in dom(h)$, $R^{\mathfrak{M}}xy$ iff $R^{\mathfrak{N}}h(x)h(y)$.

Generated back-and-forth systems Let $\mathfrak{M}$ and $\mathfrak{N}$ be two first-order models in the hybrid signature. A generated back-and-forth system between $\mathfrak{M}$ and $\mathfrak{N}$ is a non-empty family F of finite partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ satisfying the following two extension rules:

($\Diamond$-extension)

- (forth) if $h \in F$, $x \in dom(h)$, and $R^{\mathfrak{M}}xy$, then $h \cup \{\langle y, y'\rangle\} \in F$ for some $y' \in N$.
- (back) if $h \in F$, $x \in rn(h)$, and $R^{\mathfrak{N}}xy$, then $h \cup \{\langle y', y\rangle\} \in F$ for some $y' \in M$.

(nominal extension)

- (forth) if $h \in F$ and there exists an $x \in M$ such that $V^{\mathfrak{M}}(i) = \{x\}$ for some nominal i, then there exists an $x' \in N$ such that $h \cup \{\langle x, x'\rangle\} \in F$.
- (back) a similar condition backwards.

If $\bar{m} \in M^k$, $\bar{n} \in N^k$, then $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$ means there is a generated back-and-forth system linking $\mathfrak{M}$ and $\mathfrak{N}$ which contains a partial isomorphism sending m_i to n_i $(0 \leq i \leq k)$.

Note how closely this definition follows the familiar one from first-order logic (cf. e.g., [Hod93]). In fact, if we think of such a system as describing an Ehrenfeucht-Fraïssé game, then the sole difference is that in the "generated back-and-forth game" the universal player must choose his moves from R-successors or worlds named by a nominal, whereas any choice is allowed in the full first-order game. Because play is restricted to *accessible* worlds, generated back-and-forth systems are linked to the modal notion of generated submodels.

Definition 3.6 (Generated Submodel) *Let $\mathfrak{M} = \langle M, R, V\rangle$ be a hybrid model and $S \subseteq M$. Let* NAMED *denote the subset of M whose elements are in the interpretation of some nominal. The* submodel of $\mathfrak{M}$ generated by S *is the substructure of $\mathfrak{M}$ with domain $\{m \in M \mid \exists s \in S \cup \mathsf{NAMED}.(R^*sm)\}$ (R^* is the reflexive and transitive closure of R). This is also called the S-generated submodel of $\mathfrak{M}$.*

Note that if we work in an ordinary (non-hybrid) language, $\mathsf{NAMED} = \emptyset$, and we have the familiar modal notion of a generated submodel; and that if in addition S is a singleton set, we have the usual modal notion of a point-generated (or rooted) submodel.

We now define two notions of invariance. The first is taken from [vB83]. A first-order formula $\varphi(\bar{x})$ in free variables $\bar{x}$ in a signature with one binary relation R, unary predicates and constants (and equality) is *invariant for generated submodels* if for all models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{M}', \bar{m})$ such that $\mathfrak{M}'$ is the $\bar{m}$-generated submodel of $\mathfrak{M}$,

$$\mathfrak{M} \models \varphi[\bar{m}] \text{ if and only if } \mathfrak{M}' \models \varphi[\bar{m}].$$

Similarly, we say that a first-order formula $\varphi(\bar{x})$ in the same signature is *invariant for generated back-and-forth systems* if for all models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$, $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$ implies

$$\mathfrak{M} \models \varphi[\bar{m}] \text{ if and only if } \mathfrak{N} \models \varphi[\bar{n}].$$

Theorem 3.7 *Let $\varphi(\bar{x})$ be a first-order formula in the hybrid signature. Then the following are equivalent:*

i. $\varphi(\bar{x})$ *is equivalent to a formula in the bounded fragment.*

ii. $\varphi(\bar{x})$ *is invariant for generated submodels.*

iii. $\varphi(\bar{x})$ *is invariant for generated back-and-forth systems.*

PROOF.

i. $\Rightarrow$ *ii.* is obvious.

ii. $\Rightarrow$ *iii.* First note that $\varphi(\bar{x})$ is invariant for generated submodels if and only if $\neg\varphi(\bar{x})$ is. Now, suppose $\varphi(\bar{x})$ is invariant for generated submodels but *not* preserved under generated back-and-forth systems. Then we have models $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$, a generated back-and-forth system linking $\bar{m}$ and $\bar{n}$, and $\mathfrak{M} \models \varphi[\bar{m}]$ while $\mathfrak{N} \models \neg\varphi[\bar{n}]$.

Let $\mathfrak{M}'$ ($\mathfrak{N}'$) be the $\bar{m}$- ($\bar{n}$-) generated submodel of $\mathfrak{M}$ ($\mathfrak{N}$). Then still $\mathfrak{M}' \models \varphi[\bar{m}]$ and $\mathfrak{N}' \models \neg\varphi[\bar{n}]$ by invariance, and clearly $(\mathfrak{M}', \bar{m}) \equiv_R (\mathfrak{N}', \bar{n})$. But then $(\mathfrak{M}', \bar{m})$ and $(\mathfrak{N}', \bar{n})$ have the same first-order theory by the following argument. Because $(\mathfrak{M}', \bar{m}) \equiv_R (\mathfrak{N}', \bar{n})$, $\exists$loise (the existential player) has a winning strategy in all games where $\forall$belard (the universal player) only plays *immediate* R-successors or points named by a nominal. But since the models are generated, if they played the classic Ehrenfeucht-Fraïssé game instead, he could only play worlds which are accessible by a finite R-path from either the root or one of the named worlds. This means she has a winning strategy for the classic Ehrenfeucht-Fraïssé game too, contradicting the claim that $\mathfrak{M} \models \varphi[\bar{m}]$ and $\mathfrak{N}' \models \neg\varphi[\bar{n}]$.

iii. $\Rightarrow$ *i.* A fairly standard diagram-chasing argument (cf. e.g., [vB96]). Let $\varphi(\bar{x})$ be as in the hypothesis and $BC(\varphi(\bar{x}))$ be the bounded consequences of $\varphi(\bar{x})$ (that is, the consequences of $\varphi(\bar{x})$ that belong to the bounded fragment). We will show that $BC(\varphi(\bar{x})) \models \varphi(\bar{x})$, from which the result follows by compactness. (In this notation we interpret the $\bar{x}$ as constants, or equivalently we use the local version of first-order consequence, cf. [End72].)

If $BC(\varphi(\bar{x}))$ is inconsistent we are done. Otherwise, let $(\mathfrak{M}, \bar{m})$ be a model of $BC(\varphi(\bar{x}))$ and $(\mathfrak{N}, \bar{n})$ be a model of $\varphi(\bar{x})$ together with the bounded theory of $(\mathfrak{M}, \bar{m})$. (Such a model can easily be shown to exist.) Take ω-saturated extensions $(\mathfrak{M}^+, \bar{m})$ and $(\mathfrak{N}^+, \bar{n})$. Create a family F of finite functions between M^+ and N^+ as follows: $f : \bar{x} \to \bar{y}$ is in F iff $(\mathfrak{M}^+, \bar{x})$ and $(\mathfrak{N}^+, \bar{y})$ make the same bounded formulas true. It is easy to show that F is a generated back and forth system linking $\bar{m}$ and $\bar{n}$. Now we can start diagram chasing: $\mathfrak{N} \models \varphi[\bar{n}]$ then (by elementary extension) $\mathfrak{N}^+ \models \varphi[\bar{n}]$, then (by invariance) $\mathfrak{M}^+ \models \varphi[\bar{m}]$, then (passing to an elementary submodel) $\mathfrak{M} \models \varphi[\bar{m}]$ as desired. QED

3.3 Hybrid bisimulations

We have just seen that by weakening the notion of an Ehrenfeucht-Fraïssé game we can link the bounded fragment (and hence $\mathcal{H}(\downarrow, @)$) with generated submodels. But in spite of its binding apparatus, $\mathcal{H}(\downarrow, @)$ has a distinctly modal flavor. Is it not also possible to strengthen the notion of *bisimulation* (the standard notion of equivalence between models in modal logic) with clauses for $\downarrow$ and $@$, and so characterize $\mathcal{H}(\downarrow, @)$ in intrinsically modal terms? That's what we will do in this section. The approach has an advantage over the use of generated back-and-forth systems: preservation results can be easily obtained for reducts as well.

Recall that for ordinary propositional modal logics, bisimulations are non-empty binary relations linking the domains of models, with the restriction that only worlds with identical atomic information and matching accessibility relations should be connected (see [vB83, Definition 3.7]; here bisimulations are called p-relations). Now, if we want to extend this notion to $\mathcal{H}(\downarrow, @)$, we need to take care of assignments to world variables as well. To this end, hybrid bisimulations will not simply link worlds, rather they will link pairs $(\bar{m}, m)$, where m is a world and $\bar{m}$ is an assignment. We start by defining k-bisimulations, which are the correct notion of bisimulation for formulas φ such that $\mathsf{WVAR}(\varphi) \subseteq \{x_1, \ldots, x_k\}$.

k-bisimulation. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. Let $\overset{k}{\sim}$ be a binary relation between ${}^kM \times M$ and ${}^kN \times N$. So $\overset{k}{\sim}$ relates tuples $((m_1, \ldots, m_k), m)$ with tuples $((n_1, \ldots, n_k), n)$. We write these tuples as $(\bar{m}, m)$. Notice that $\bar{m}$ can be seen as an assignment over $(x_1, \ldots, x_k)$. A non-empty relation $\overset{k}{\sim}$ is called a *k-bisimulation* if it satisfies the following properties

(prop) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then $m \in V^{\mathfrak{M}}(a)$ iff $n \in V^{\mathfrak{N}}(a)$, for $a \in \mathsf{PROP} \cup \mathsf{NOM}$.

(var) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for all $j \leq k$, $m_j = m$ iff $n_j = n$.

(forth) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$ and $R^{\mathfrak{M}}mm'$, then there exists an $n' \in N$ such that $R^{\mathfrak{N}}nn'$ and $(\bar{m}, m') \overset{k}{\sim} (\bar{n}, n')$.

(back) A similar condition from $\mathfrak{N}$ to $\mathfrak{M}$.

(@) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for every nominal $i \in \mathsf{NOM}$, if $m' \in V^{\mathfrak{M}}(i)$ and $n' \in V^{\mathfrak{N}}(i)$ then $(\bar{m}, m') \overset{k}{\sim} (\bar{n}, n')$, and for every $j \leq k$, $(\bar{m}, m_j) \overset{k}{\sim} (\bar{n}, n_j)$.

(↓) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then for every $j \leq k$, $(\bar{m}^{x_j}_m, m) \overset{k}{\sim} (\bar{n}^{x_j}_n, n)$.

Note that since $\downarrow$ and $@$ are self-dual, we can collapse the back and forth clauses for these modalities into one. We write $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$ if there exists a k-bisimulation between the two models.

To extend the notion to the full language we need to add only one further condition.

ω-**bisimulation.** Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models. An ω-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ is a non-empty family of k-bisimulations satisfying the following *storage rule*:

(sto) If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then $(\bar{m} * m, m) \overset{k+1}{\sim} (\bar{n} * n, n)$.

Here and elsewhere, $\bar{m} * m$ denotes the tuple obtained from concatenating $\bar{m}$ and m. Let $\bar{m}$ ($\bar{n}$) be an M-tuple (N-tuple). Then $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$ means that there exists an ω-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ such that $(\bar{m}, \bar{m}(0)) \overset{k}{\sim} (\bar{n}, \bar{n}(0))$, for k the length of $\bar{m}$.

Some remarks. First, k and ω-bisimulations can be restricted to a given set of propositional variables and nominals $\mathsf{PROP} \cup \mathsf{NOM}$ by restricting **(prop)** and **(@)** accordingly. Second, the modular definition of k and ω-bisimulation will lead to results for reducts of the language as well. For instance if we delete $\downarrow$ from the language, we just delete the **(↓)** clause from the definition of bisimulation and we obtain the appropriate notion for $\mathcal{H}(@)$. Of course, if we delete the variables from the language, we don't need the assignment tuples anymore, and the bisimulation becomes just a relation between worlds, as usual. Then for the language without $\downarrow, @$ and variables, the standard definition of bisimulation applies (the condition **(prop)** takes care of the nominals). If we add $@$ to this language, we just have to add the following clause

(@′) For all nominals i, if $V^{\mathfrak{M}}(i) = \{m\}$ and $V^{\mathfrak{N}}(i) = \{n\}$, then $m \sim n$.

Preservation results for all these alternatives can be given (the required proofs follow much the same lines as the proofs below) and we shall prove one such result in Section 6.

The first important fact about hybrid bisimulations is that they preserve truth:

Proposition 3.8 *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two hybrid models, $m \in M$, $n \in N$. Then,*

i. If $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$, with $\overset{k}{\sim}$ over a given set $\mathsf{PROP} \cup \mathsf{NOM}$, *then for all formulas φ over the signature* $\langle \mathsf{PROP}, \mathsf{NOM}, \{x_1, \ldots, x_k\} \rangle$, $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$ *implies* $\mathfrak{M}, \bar{m}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, \bar{n}, n \Vdash \varphi$.

ii. If $(\mathfrak{M}, m) \overset{\omega}{\sim} (\mathfrak{N}, n)$, with $\overset{\omega}{\sim}$ over a given set $\mathsf{PROP} \cup \mathsf{NOM}$, *then for all* sentences φ *over the signature* $\langle \mathsf{PROP}, \mathsf{NOM}, \mathsf{WVAR} \rangle$, $\mathfrak{M}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, n \Vdash \varphi$. *(Recall that for sentences the choice of assignment is irrelevant.)*

PROOF.

i. By a straightforward inductive argument.

ii. Let $(\mathfrak{M}, m) \overset{\omega}{\sim} (\mathfrak{N}, n)$ and let φ be a hybrid sentence. Then it contains variables (after renaming) say $\{x_1, \ldots, x_k\}$. We have $(\langle m \rangle, m) \overset{1}{\sim} (\langle n \rangle, n)$, so $k - 1$ applications of the storage rule gives us $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, where $\bar{m}$ is a k-tuple consisting of m's and similarly for $\bar{n}$. But then, by *i.*, $\mathfrak{M}, \bar{m}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, \bar{n}, n \Vdash \varphi$, whence since φ is a sentence $\mathfrak{M}, m \Vdash \varphi \Leftrightarrow \mathfrak{N}, n \Vdash \varphi$. QED

The notion of k-bisimulation has a distinct modal flavor. But a very first-order notion is hidden inside: partial isomorphism.

Proposition 3.9 *Let $k \geq 2$, and let $\mathfrak{M} \overset{k}{\sim} \mathfrak{N}$. If $(\bar{m}, m) \overset{k}{\sim} (\bar{n}, n)$, then the function f defined as $f(m) = n$ and $f(m(i)) = n(i)$ is a partial isomorphism between $\{m(1), \ldots, m(k), m\}$ and $\{n(1), \ldots, n(k), n\}$.*

PROOF. The map f is a bijection by **(var)** and **(@)**. By **(prop)** and **(@)**, f preserves nominals and propositional variables. To see that it preserves the accessibility relation suppose $R^{\mathfrak{M}}xy$. There are three cases.

(Case 1: $x = m,\ y = m_i$.) Then by **(forth)** there exists an n' such that $R^{\mathfrak{N}}nn'$ and $(\bar{m}, m_i) \overset{k}{\sim} (\bar{n}, n')$. But $\bar{m}(i) = m_i$, so by **(var)**, $n' = \bar{n}(i)$, whence $R^{\mathfrak{N}}nf(m(i))$.

(Case 2: $x = m_i,\ y = m$.) Let $j \neq i$. Such a j exists because we assumed that $k \geq 2$. By **(↓)**, $(\bar{m}^j_m, m) \overset{k}{\sim} (\bar{n}^j_n, n)$. Then by **(@)**, $(\bar{m}^j_m, m_i) \overset{k}{\sim} (\bar{n}^j_n, n_i)$. Now continue as in case 1.

(Case 3: $x = m_i,\ y = m_j$.) By **(@)**, $(\bar{m}, m_i) \overset{k}{\sim} (\bar{n}, n_i)$. Now continue as in case 1.
Thus $R^{\mathfrak{M}}xy$ implies $R^{\mathfrak{N}}f(x)f(y)$. For the other direction use **(back)** in the same way. QED

Note that the condition $k \geq 2$ is crucial. We use it together with **(↓)** to store the information about m. In a model where Rm_im holds, we have $\bar{m}, m \Vdash {\downarrow} x_j.@_{x_i}\Diamond x_j$.

Thus there is a clear link between our earlier work on generated back-and-forth systems, and the next theorem shouldn't come as a surprise:

Theorem 3.10 *Let $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$ be two models. Then the following are equivalent*

i. $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$.

ii. $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$.

PROOF.
i. ⇒ *ii.* Let $(\mathfrak{M}, \bar{m}) \overset{\omega}{\sim} (\mathfrak{N}, \bar{n})$. Define a family F of maps as follows: $f \in F$ if there exists $(\bar{x}, x') \overset{k}{\sim} (\bar{y}, y')$ and f is defined as in Proposition 3.9.

Clearly $\bar{m}$ and $\bar{n}$ are connected by a map. By Proposition 3.9 all maps are partial isomorphisms. We show the forth side of **(nominal extension)**; all other conditions have similar proofs. Suppose $f \in F$ and $\mathbf{z} \in M$ and $V^{\mathfrak{M}}(i) = \{\mathbf{z}\}$, for some nominal i. Then for some $\bar{x}$, x, $\bar{y}$, y, $(\bar{x}, x') \overset{k}{\sim} (\bar{y}, y')$ by definition of F. Then $(\bar{x}*x', x') \overset{k+1}{\sim} (\bar{y}*y', y')$ by **(sto)**. But then by **(@)**, $(\bar{x}*x', \mathbf{z}) \overset{k+1}{\sim} (\bar{y}*y', \mathbf{z}')$ for $V^{\mathfrak{N}}(i) = \{\mathbf{z}'\}$. Thus the wanted extension is in F.

ii. ⇒ *i.* Let $(\mathfrak{M}, \bar{m}) \equiv_R (\mathfrak{N}, \bar{n})$. We define the following family of relations: for any $f \in F$, for any k, for any tuple $\bar{m}$ in the k-th power of the domain of f and for any m in the domain of f, we set $(\bar{m}, m) \overset{k}{\sim} (f\bar{m}, f(m))$. It is easy to check that this is an ω-bisimulation. QED

It is possible to prove a direct characterization result for $\mathcal{H}(\downarrow, @)$ in terms of invariance for k-bisimulations, using again a diagram chasing argument. We are not going to do this here since in the next section we shall take a detour via the bounded fragment to reach the same result. It is also possible to develop k-pebble versions of generated back-and-forth systems; this notion takes the exact number of variables used in formulas into account. It is not difficult to see that $k+1$-pebble generated back-and-forth systems correspond to k-bisimulations.

3.4 Harvest

It is time to draw together the threads we have developed. First we note their consequences for $\mathcal{H}(\downarrow, @)$ expressivity over models. Then we note the consequences for frames and what this tells us about hybrid completeness. Finally we discuss hybrid *tense* logic.

3.4.1 Expressivity over models

We have the following five-fold characterization of $\mathcal{H}(\downarrow, @)$:

Theorem 3.11 *Let $\varphi(\bar{x})$ be a first-order formula in the hybrid signature (with equality). Then the following are equivalent*

- *i. $\varphi(\bar{x})$ is equivalent to the standard translation of a $\mathcal{H}(\downarrow, @)$ formula.*
- *ii. $\varphi(\bar{x})$ is invariant for generated submodels.*
- *iii. $\varphi(\bar{x})$ is invariant for generated back-and-forth systems.*
- *iv. $\varphi(\bar{x})$ is invariant for ω-bisimulation.*
- *v. $\varphi(\bar{x})$ is equivalent to a formula in the bounded fragment of first-order logic.*

PROOF. By Corollary 3.4, Theorem 3.7, Proposition 3.8, and Theorem 3.10. QED

But these have obvious consequences for the ordinary modal correspondence language. In particular, if we consider *nominal-free* hybrid sentences, then we obtain a five-fold characterization of the fragment of first-order logic in the classical modal signature which is invariant for generated submodels:

Corollary 3.12 *Let $\varphi(x)$ be a first-order formula in the modal signature with equality. Then the following are equivalent*

- *i. $\varphi(x)$ is equivalent to the standard translation of a nominal-free $\mathcal{H}(\downarrow, @)$ sentence.*
- *ii. $\varphi(x)$ is invariant for generated submodels (now in the standard modal sense).*
- *iii. $\varphi(x)$ is invariant for R-generated back-and-forth systems, where an R-generated back-and-forth system is a back-and-forth system satisfying only the* $\Diamond$**-extension** *rule.*
- *iv. $\varphi(x)$ is invariant for ω-bisimulation.*
- *v. $\varphi(x)$ is equivalent to a formula in the bounded fragment of first-order logic without constants.*

3.4.2 Frames and completeness

Since the late 1950s, a central theme in modal logic has been linking modal formulas with properties of frames and investigating when they give rise to complete axiomatizations for the frame classes they define. The work of the previous section yields a characterization of the frame-defining abilities of pure nominal-free sentences. Moreover, the axiomatic investigations of [BT98a, BT99] (and indeed, the tableaux-based investigations of [Bla00]) show that there is a perfect match between definability and completeness for pure nominal-free sentences. By combining these results we obtain matching definability and completeness results for a wide range of first-order definable frame classes.

In modal correspondence theory, the first-order language (with equality) over the signature consisting simply of a binary symbol R is called the (first-order) *frame language*. We shall call a formula φ in the frame language containing exactly one free variable a *frame condition*. The class of frames defined by a frame condition $\varphi(x)$ is the class in which the universal closure $\forall x.\varphi(x)$ is true; we call this class $\textsc{frames}(\forall x.\varphi(x))$.

Before proceeding further, two simple observations are in order. First, note that if we apply the standard translation ST to a pure nominal-free sentence α , then $ST(\alpha)$ is a frame condition with free-variable x. Furthermore, note that for any frame $\mathfrak{F} = \langle W, R \rangle$ we have that $\mathfrak{F} \Vdash \alpha$ iff $\mathfrak{F} \Vdash \forall x.ST(\alpha)$; this is an immediate consequence of the definition of frame validity.

Theorem 3.13 *Let $\mathsf{K}[\mathcal{H}(\downarrow, @)]$ be the axiomatization given in Section 2, and for any hybrid sentence α let $\mathsf{K}[\mathcal{H}(\downarrow, @)] + \alpha$ be the system obtained by adding α as an additional axiom. Then, if $\varphi(x)$ is a frame condition and $\varphi(x)$ is invariant under generated submodels (in the usual modal sense) we have that:*

i. If $\varphi(x)$ is in the bounded fragment, then the pure nominal free sentence $\downarrow x.HT(\varphi(x))$ defines FRAMES$(\forall x.\varphi(x))$*. Moreover, $\mathsf{K}[\mathcal{H}(\downarrow, @)]+ \downarrow x.HT(\varphi(x))$ is strongly complete with respect to* FRAMES$(\forall x.\varphi(x))$.

ii. If $\varphi(x)$ is not in the bounded fragment, there is a pure nominal free sentence α such that α defines FRAMES$(\forall x.\varphi(x))$*, and $ST(\alpha)$ is equivalent to $\varphi(x)$. Moreover, $\mathsf{K}[\mathcal{H}(\downarrow, @)] + \alpha$ is strongly complete with respect to* FRAMES$(\forall x.\varphi(x))$.

Conversely, if α is pure nominal-free sentence, then α defines FRAMES$(\forall x.ST(\alpha(x)))$*, and $\mathsf{K}[\mathcal{H}(\downarrow, @)] + \alpha$ is complete with respect to* FRAMES$(\forall x.ST(\alpha(x)))$.

PROOF. The converse condition was proved in [BT98b], so let's examine the other direction.

For item *i.*, we first remark that as $\varphi(x)$ belongs to the *frame* language, it contains no unary predicate symbols, hence $HT(\varphi(x))$ is a *pure* formula; that $\downarrow x.HT(\varphi(x))$ is a pure nominal-free sentence is thus clear. Now, by Corollary 3.3, for any model $\mathfrak{M} = (\mathfrak{F}, V)$ and any $m \in M$,

$$(\mathfrak{F}, V) \models \varphi(m) \text{ iff } (\mathfrak{F}, V), m \Vdash \downarrow x.HT(\varphi).$$

But this means that

$$(\mathfrak{F}, V) \models \forall x.\varphi \text{ iff } (\mathfrak{F}, V) \Vdash \downarrow x.HT(\varphi).$$

But as $\varphi(x)$ contains no unary predicate symbols (and $\downarrow x.HT(\varphi)$ no propositional variables) V is irrelevant, and hence

$$\mathfrak{F} \models \forall x.\varphi(x) \text{ iff } \mathfrak{F} \Vdash \downarrow x.HT(\varphi).$$

But this means that $\downarrow x.HT(\varphi(x))$ defines FRAMES$(\forall x.\varphi(x))$. Completeness follows using the arguments of [BT98b].

For item *ii.*, we know that $\varphi(x)$ being invariant under generated submodels is equivalent to a formula in the bounded fragment — but is it equivalent to a *frame condition* $\varphi'(x)$? In fact, this can be established by modifying the diagram chasing argument used in the proof of Theorem 3.7. The key point to observe is that instead of showing that *BC*$(\varphi(x)) \models \varphi(x)$, we can show by the same method that *FC*$(\varphi(x)) \models \varphi(x)$, where *FC* are all the frame conditions implied by $\varphi(x)$. Thus there is an equivalent frame condition $\varphi'(x)$, and we can take α to be $\downarrow x.HT(\varphi'(x))$. The remainder of the proof is as for item *i*. QED

3.4.3 Hybrid tense logic

The characterization results have a particularly natural interpretation in the setting of hybrid *tense* logic. Recall that in tense logic we write $\Box$ as G, $\Diamond$ as F, and that we also have at our disposal an operator H (a $\Box$-operator that scans the *converse* of the accessibility relation) and its dual P (a

$\Diamond$-operator that scans the converse of the accessibility relation). It is straightforward to hybridize tense logic by adding nominals, @ and $\downarrow$ (though now it is more natural to talk of *time* variables rather than world variables) thus forming the language $\mathcal{H}_t(\downarrow, @)$. To cope with the backward looking operators, we need a slightly more liberal notion of generated submodel: a point t belongs to the submodel *temporally generated* by a subset S if t is reachable from some point $s \in S$ by making a finite sequence of moves through the accessibility relation, where both forward *and backward* steps are allowed. The characterization results we have proved hold for $\mathcal{H}_t(\downarrow, @)$ under this notion of generated submodel.

But let's press matters a little further. Note that in *nominal-free sentences* of $\mathcal{H}_t(\downarrow, @)$, all occurrences of @ are eliminable. As a simple example, consider the definition of the *Until* operator:

$$\mathit{Until}(\varphi, \psi) \ := {\downarrow} x.\mathsf{F} {\downarrow} y.@_x(\mathsf{F}(y \wedge \varphi) \wedge \mathsf{G}(\mathsf{F}y \rightarrow \psi)).$$

(This is simply the definition given in Section 2 written in tense logical notation.) But observe that the following nominal-free sentence has the same effect:

$$\mathit{Until}(\varphi, \psi) \ := {\downarrow} x.\mathsf{F} {\downarrow} y.\mathsf{P}(x \wedge (\mathsf{F}(y \wedge \varphi) \wedge \mathsf{G}(\mathsf{F}y \rightarrow \psi))).$$

That is, instead of retrieving the point named x using the @ operator, we can "reach back" for this point using P.

This observation (first made in [BT98a]) is completely general. As long as a $\mathcal{H}_t(\downarrow, @)$ formula doesn't contain nominals or free time variables, it will always be possible to simulate @ by zig-zagging back to the binding point using the tense operators. More precisely, suppose a nominal free sentence φ has a temporal depth of n (that is, the maximal depth of embedding of tense operators is n) and that φ is satisfied at a time m. Then when we evaluate a subformula of φ of the form $@_x\psi$ at some point m' (note that m' cannot be more than n forward and back steps from m) then we know that x must bound to a point m'' (which is also not more than n forward and back steps from m). Hence m' and m'' are separated by at most $2n$ (forward and back) steps. We can define an operator $@^{2n}$ that allows us to zig-zag to a named time lying within $2n$ steps as follows. Let $\mathrm{ZZ}2n$ be the set of all non-empty finite sequences of F and P operators of length at most $2n$. Then for any formula ψ and any variable x we define:

$$@^{2n}_x\psi \ := \ (x \wedge \psi) \vee \bigvee_{Z \in \mathrm{ZZ}2n} Z(x \wedge \psi).$$

Hence, given a nominal free sentence φ of temporal depth n, we eliminate all occurrences of @ as follows. Let $@_x\psi$ be a subformula of φ where ψ contains no occurrences of @. Replace $@_x\psi$ by $@^{2n}_x\psi$ to form φ'. Repeating this procedure (starting with φ') produces an equivalent nominal-free sentence containing no occurrences of @. Thus, in the setting of tense logic, our characterization results for nominal free sentences go through without the help of @.

This is a pleasing result, for there are also non-technical reasons for viewing $\mathcal{H}_t(\downarrow)$ as a key system: this language can be viewed as a marriage between the ideas of Arthur Prior and Hans Reichenbach.

That $\mathcal{H}_t(\downarrow)$ captures ideas from Arthur Prior is clear: he invented tense logic precisely to capture the "internal" perspective on time which underlies temporal discourse in natural language (see in particular [Pri67]). But while the Priorean perspective gets a lot right, it misses a crucial fact about temporal discourse: tenses are very often *referential*. That is, tenses in natural language often achieve their effect by referring to specific points of time, and many semantic distinctions between natural language tenses cannot be drawn without taking referential effects into account. The importance of temporal reference was first made clear in the work of Hans Reichenbach, and many modern theories of tense (for example, [Com85]) adopt a fundamentally Reichenbachian stance.

It should be clear where this is heading: $\downarrow$ can be seen as the Reichenbachian device *par excellence*. In a sense, $\downarrow$ gives us a sort of *generalized present tense*; it enables us to "store" an evaluation point, thereby making it possible to insist later that certain events happened at *that* time, or that certain other events must be viewed from that particular perspective. This is precisely the kind of expressive power we need to encode Reichenbach's ideas. And crucially, the use of $\downarrow$ does not in any sense conflict with Prior's use of tense operators. Quite the reverse: F, P, and $\downarrow$ work together beautifully. No auxiliary apparatus (not even @) is required to blend the two approaches, and the result is a language which exactly captures first-order temporal reachability.

4 Interpolation

In this section we show that $\mathcal{H}(\downarrow, @)$ is well behaved in yet another sense: it has the interpolation property. Interpolation is a much studied notion. Originally considered a property of deductive systems, it was proved for first-order logic in [Cra57] using a proof theoretic argument. We shall view interpolation as a property of *consequence relations* and will prove it using semantic arguments (as is done, for example, in [CK90]). Incidentally, Jerry Seligman has recently announced a proof-theoretic proof of interpolation for $\mathcal{H}(\downarrow, @)$.

Before plunging into the details, note that in modal logic we can distinguish between *strong arrow interpolation* (AIP) and *weak turnstile interpolation* (TIP). AIP implies TIP, but not conversely. We will prove AIP for $\mathcal{H}(\downarrow, @)$, disprove AIP and TIP for its finite variable fragments (our earlier work on k-bisimulations will enable us to construct straightforward counterexamples) and show that TIP holds for the sublanguage $\mathcal{H}(@)$. Here are the definitions of these concepts:

AIP A logic L has the *Arrow Interpolation Property* (AIP) if, whenever $\models_L \varphi \to \psi$, there exists a formula θ such that $\models_L \varphi \to \theta$, $\models_L \theta \to \psi$ and $I\!\!P(\theta) \subseteq I\!\!P(\varphi) \cap I\!\!P(\psi)$.

TIP A logic L has the *Turnstile Interpolation Property* (TIP) if, whenever $\varphi \models_L \psi$, there exists a formula θ such that $\varphi \models_L \theta$, $\theta \models_L \psi$ and $I\!\!P(\theta) \subseteq I\!\!P(\varphi) \cap I\!\!P(\psi)$.

For first-order logic these notions are equivalent, but in modal logic this is not the case (as we see below, equivalence depends on both compactness and the availability of a deduction theorem; cf. also [Cze82]). Further, note that the meaning of TIP depends on the way we define the consequence relation $\varphi \models \psi$. In Section 2 we introduced two consequence relations: local consequence and global consequence (see [vB83] or [MV97] for a discussion of their relative merits). So there are two plausible definitions of the turnstile interpolation property, TIP^{loc} and TIP^{glo}.

In modal logic these different notions of interpolation are related as follows:

Proposition 4.1

i. AIP and TIP^{loc} *are equivalent.*

ii. If the local consequence relation is compact, then AIP implies TIP^{glo}*.*

For this reason, from now on we take TIP to be defined using the *global* consequence relation. AIP and TIP are often referred to as the strong and weak interpolation properties respectively, and we shall sometimes use this terminology.

We turn to the technicalities of the interpolation result. As is usual in interpolation proofs, where language related issues require special care, we replace the notion of *consistency*

Definition 4.2 (Consistency) *Let T be a set of formulas in $\mathcal{L}$. Then T is* consistent *iff there is a model $\mathfrak{M}$, an $m \in M$, and an assignment g such that for all $\varphi \in T$, $\mathfrak{M}, g, m \Vdash \varphi$.*

by the finer-grained notion of *separability*

Definition 4.3 (Separability) *Let T, U, L be sets of formulas in $\mathcal{L}$. We say that the pair $\langle T, U\rangle$* is separable with respect to L *if there exists a formula $\theta \in L$ such that $T \models \theta$ and $U \models \neg\theta$. $\langle T, U\rangle$* is inseparable with respect to L *if it is not separable with respect to L.*

We are ready to prove the main result of this section.

Theorem 4.4 (Arrow Interpolation for $\mathcal{H}(\downarrow, @)$) *Let φ and ψ be formulas in $\mathcal{L}$ such that $\models \varphi \to \psi$. Then there exists a formula θ such that*

i. $\models \varphi \to \theta$ *and* $\models \theta \to \psi$.

ii. $I\!P(\theta) \subseteq I\!P(\varphi) \cap I\!P(\psi)$.

PROOF. Suppose we are given formulas φ_0 and ψ_0 such that there is no interpolant for $\varphi_0 \to \psi_0$. We will prove that $\not\models \varphi_0 \to \psi_0$ by producing a model $\mathfrak{M} = \langle M, R, V\rangle$ and an assignment g such that for some $m \in M$, $\mathfrak{M}, g, m \Vdash \varphi_0 \wedge \neg\psi_0$. The proof uses the method of [CK90], in which two related models are simultaneously built using fresh constants, with nominals playing the role of constants.

We can assume that $\{\varphi_0\}$ and $\{\neg\psi_0\}$ are consistent (for if they are not, then either $\bot$ or $\top$ is an interpolant). Furthermore they must be inseparable over the formulas in $\mathcal{L}$ with propositional variables and nominals in $I\!P(\varphi_0) \cap I\!P(\psi_0)$.

Let $\mathcal{L}'$ be the hybrid language over the signature $\langle \mathsf{PROP}, \mathsf{NOM} \cup \mathrm{N}, \mathsf{WVAR}\rangle$, where $\mathrm{N} = \{n_0, \ldots, n_k, \ldots\}$ is a countably infinite set of new nominals. For any formula φ define the restricted language $\mathcal{L}_\varphi$ as $\{\xi \in \mathcal{L} \mid I\!P(\xi) \subseteq I\!P(\varphi)\}$ and $\mathcal{L}'_\varphi$ as $\{\xi \in \mathcal{L}' \mid I\!P(\xi) \subseteq I\!P(\varphi) \cup \mathrm{N}\}$. Let $\mathcal{L}'_{\varphi_0\psi_0} = \mathcal{L}'_{\varphi_0} \cap \mathcal{L}'_{\psi_0}$.

Let $\varphi_1, \ldots, \varphi_k, \ldots$ be an enumeration of all formulas in $\mathcal{L}'_{\varphi_0}$,
$\psi_1, \ldots, \psi_k, \ldots$ be an enumeration of all formulas in $\mathcal{L}'_{\psi_0}$.

We define the sequences, $\{n_0\} \cup \{\varphi_0\} = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$
$\{n_0\} \cup \{\neg\psi_0\} = U_0 \subseteq U_1 \subseteq U_2 \subseteq \ldots$
as follows:

- If $T_j \cup \{\varphi_j\}$ and U_j are separable over $\mathcal{L}'_{\varphi_0\psi_0}$ then $T_{j+1} = T_j$, else
  - if $\varphi_j \neq @_s s$ and $\varphi_j \neq @_s\Diamond\varphi'$ for $s \in \mathsf{WSYM}$, then $T_{j+1} = T_j \cup \{\varphi_j\}$,
  - if $\varphi_j = @_s s$, then $T_{j+1} = T_j \cup \{\varphi_j\} \cup \{@_s(n_k \wedge s)\}$, for $n_k \in \mathrm{N}\backslash\mathsf{NOM}(T_j \cup U_j)$,
  - if $\varphi_j = @_s\Diamond\varphi'$, then $T_{j+1} = T_j \cup \{\varphi_j\} \cup \{@_s\Diamond(n_k \wedge \varphi')\}$, for $n_k \in \mathrm{N}\backslash\mathsf{NOM}(T_j \cup U_j \cup \{\varphi_j\})$.
- If T_{j+1} and $U_j \cup \{\psi_j\}$ are separable over $\mathcal{L}'_{\varphi_0\psi_0}$ then $U_{j+1} = U_j$, else
  - if $\psi_j \neq @_s s$ and $\psi_j \neq @_s\Diamond\psi'$ for $s \in \mathsf{WSYM}$, then $U_{j+1} = U_j \cup \{\psi_j\}$,
  - if $\psi_j = @_s s$, then $U_{j+1} = U_j \cup \{\psi_j\} \cup \{@_s(n_k \wedge s)\}$, for $n_k \in \mathrm{N}\backslash\mathsf{NOM}(T_{j+1} \cup U_j)$,
  - if $\psi_j = @_s\Diamond\psi'$, then $T_{j+1} = T_j \cup \{\psi_j\} \cup \{@_s\Diamond(n_k \wedge \psi')\}$, for $n_k \in \mathrm{N}\backslash\mathsf{NOM}(T_{j+1} \cup U_j \cup \{\psi_j\})$.

The fresh nominals play the same role as *Henkin witnesses* in first-order proofs: they ensure that we obtain models in which every world has a name. Define

$$T_\omega = \bigcup_{j\in\omega} T_j \quad \text{and} \quad U_\omega = \bigcup_{j\in\omega} U_j.$$

Claim 1 *For all $j \in \omega$, $\langle T_j, U_j\rangle$ is inseparable with respect to $\mathcal{L}'_{\varphi_0\psi_0}$. Whence $\langle T_\omega, U_\omega\rangle$ is an inseparable pair with respect to $\mathcal{L}'_{\varphi_0\psi_0}$. Furthermore T_ω (resp. U_ω) is maximal consistent in $\mathcal{L}'_{\varphi_0}$ (resp. $\mathcal{L}'_{\psi_0}$). In particular, for all $\theta \in \mathcal{L}'_{\varphi_0\psi_0}$: $\theta \in T_\omega \Leftrightarrow \theta \in U_\omega$.*

PROOF OF CLAIM. The proof is by induction on j. Separability/inseparability below is with respect to $\mathcal{L}'_{\varphi_0\psi_0}$ except when otherwise mentioned.

BASE CASE $j = 0$. Suppose $\langle T_0, U_0\rangle$ is separable. Then there is a formula $\theta \in \mathcal{L}'_{\varphi_0\psi_0}$ such that $\models n_0 \wedge \varphi_0 \to \theta$ and $\models n_0 \wedge \neg\psi_0 \to \neg\theta$. θ might contain some nominals of N, say $\{n_{i_1}, \ldots, n_{i_k}\}, k \geq 0$. Let $x_0, x_{i_1}, \ldots, x_{i_k} \in$ WVAR which don't occur in $\varphi_0, \psi_0, \theta$. We will write $\theta[x_0 x_{i_1} \ldots x_{i_k}]$ for the formula obtained from θ by replacing n_{i_j} by x_{i_j}, and n_0 by x_0. Then, making use of the complete axiomatization given in Section 2, we have:

$\models \varphi_0 \to (n_0 \to \theta)$	
$\models \downarrow x_{i_k}.(\varphi_0 \to (n_0 \to \theta[x_{i_k}]))$	Proposition 2.5
$\models \varphi_0 \to (n_0 \to \downarrow x_{i_k}.\theta[x_{i_k}])$	*Q1* twice
$\models \varphi_0 \to (n_0 \to \downarrow x_{i_1}.\ldots\downarrow x_{i_k}.\theta[x_{i_1}\ldots, x_{i_k}])$	Similarly
$\models \downarrow x_0.(\varphi_0 \to (x_0 \to \downarrow x_{i_1}.\ldots\downarrow x_{i_k}.\theta[x_0 x_{i_1}\ldots, x_{i_k}]))$	Proposition 2.5
$\models \varphi_0 \to \downarrow x_0.(x_0 \to \downarrow x_{i_1}.\ldots\downarrow x_{i_k}.\theta[x_0 x_{i_1}\ldots, x_{i_k}]))$	*Q1*
$\models \varphi_0 \to \downarrow x_0.\downarrow x_{i_1}.\ldots\downarrow x_{i_k}.\theta[x_0 x_{i_1}\ldots, x_{i_k}]$	*Q3*
$\models \neg\psi_0 \to (n_0 \to \neg\theta)$	
$\models \neg\psi_0 \to \downarrow x_0.\downarrow x_{i_1}.\ldots\downarrow x_{i_k}.\neg\theta[x_0 x_{i_1}\ldots, x_{i_k}]$	As before
$\models \neg\psi_0 \to \neg\downarrow x_0.\downarrow x_{i_1}.\ldots\downarrow x_{i_k}.\theta[x_0 x_{i_1}\ldots, x_{i_k}]$	Self Dual$_\downarrow$

$\theta[x_0 x_{i_1}\ldots, x_{i_k}]$ is a formula in $\mathcal{L}_{\varphi_0} \cap \mathcal{L}_{\psi_0}$ and thus $\langle\{\varphi_0\}, \{\neg\psi_0\}\rangle$ is separable over $\mathcal{L}_{\varphi_0} \cap \mathcal{L}_{\psi_0}$. Contradiction.

By using the inductive hypothesis "$\langle T_j, U_j\rangle$ is an inseparable pair" and going step by step through the construction, the inseparability of $\langle T_{j+1}, U_{j+1}\rangle$ is easily established. ◀

Now, to construct a model. We first recall the notion of pasted maximal consistent sets (MCS) and labeled models from [BT99]. A maximal consistent set Γ is *pasted* if

i. $@_s\varphi \in \Gamma$ implies for some nominal i, $@_s(i \wedge \varphi) \in \Gamma$ and

ii. $@_s\Diamond\varphi \in \Gamma$ implies for some nominal i, $@_s\Diamond(i \wedge \varphi) \in \Gamma$.

A pasted MCS Γ is *labeled* by a nominal i precisely when $i \in \Gamma$. Let Γ be a pasted MCS labeled by a nominal, then for all world symbols s appearing in Γ, let $\Delta_s = \{\varphi \mid @_s\varphi \in \Gamma\}$. Then the *labeled model* yielded by Γ is $\mathfrak{M} = \langle M, R, V\rangle$, where $M = \{\Delta_s \mid s \text{ is a world symbol in } \Gamma\}$, $R\Delta\Delta'$ iff $\{\varphi \mid \Box\varphi \in \Delta\} \subseteq \Delta'$ and $\Delta \in V(p)$ iff $p \in \Delta$, for p a propositional variable or nominal.

We define the natural assignment g : WVAR $\longrightarrow M$ by $g(x) = \{m \in M \mid x \in m\}$.

By construction T_ω and U_ω are pasted MCSs labeled by the nominal $n_0 \in N$. Let $\mathfrak{M}_{\varphi_0} = \langle M_{\varphi_0}, R_{\varphi_0}, V_{\varphi_0}\rangle$ be the labeled model obtained from T_ω and $\mathfrak{M}_{\psi_0} = \langle M_{\psi_0}, R_{\psi_0}, V_{\psi_0}\rangle$ the one obtained from U_ω. We denote by $\Delta_s^{\varphi_0}$ (resp. $\Delta_s^{\psi_0}$) the elements of M_{φ_0} (resp. M_{ψ_0}).

Claim 2

i. $\Delta^{\varphi_0}_{n_0} = T_\omega$ *and* $\Delta^{\psi_0}_{n_0} = U_\omega$.

ii. For $\Delta^{\varphi_0}_s \in M_{\varphi_0}$ *there is* $n \in N$ *such that* $n \in \Delta^{\varphi_0}_s$ *(or equivalently* $\Delta^{\varphi_0}_s = \Delta^{\varphi_0}_n$*). Similarly for* $\Delta^{\psi_0}_s \in \mathfrak{M}_{\psi_0}$.

PROOF OF CLAIM.
i. We show that $\Delta^{\varphi_0}_{n_0} = T_\omega$; the other case is similar. $\Delta^{\varphi_0}_{n_0}$ is an MCS because $@_{n_0}$ is self-dual. So it is sufficient to show that $\Delta^{\varphi_0}_{n_0} \supseteq T_\omega$. Let $\varphi \in T_\omega$. By *Introduction*, $\models n_0 \wedge \varphi \rightarrow @_{n_0}\varphi$. Because $@_{n_0}\varphi \in \mathcal{L}'_{\varphi_0}$, $n_0 \in T_\omega$, and T_ω is maximal in this language, $@_{n_0}\varphi \in T_\omega$. By definition $\varphi \in \Delta^{\varphi_0}_{n_0}$.
ii. By Lemma 4.3, 5) in [BT99] we have $n \in \Delta^{\varphi_0}_s \Rightarrow \Delta^{\varphi_0}_s = \Delta^{\varphi_0}_n$. We prove the case for $\mathfrak{M}_{\varphi_0}$. $@_s s$ is a formula in $\mathcal{L}'_{\varphi_0}$, hence $@_s s = \varphi_j$ for some j. As $@_s s$ is a theorem, $\{@_s s\}$ will be added to T_{j+1} together with $@_s(s \wedge n_k)$ for a new nominal n_k. It follows that $n_k \in \Delta^{\varphi_0}_s$ and hence $\Delta^{\varphi_0}_{n_k} = \Delta^{\varphi_0}_s$. ◀

From Claim 2 and results in [BT99] it follows that $\mathfrak{M}_{\varphi_0}$ and $\mathfrak{M}_{\psi_0}$ are hybrid models satisfying a Truth Lemma, and that the natural assignment g (on either of these models) really is an assignment. Thus we obtain

$$\mathfrak{M}_{\varphi_0}, g_{\varphi_0}, \Delta^{\varphi_0}_{n_0} \Vdash \varphi \text{ and } \mathfrak{M}_{\psi_0}, g_{\psi_0}, \Delta^{\psi_0}_{n_0} \Vdash \neg\psi. \tag{1}$$

Furthermore the two models are very closely related.

Claim 3 *Let a function* $h : M_{\varphi_0} \longrightarrow M_{\psi_0}$ *be defined by* $h(\Delta^{\varphi_0}_n) = \Delta^{\psi_0}_n$*, for* $n \in N$*. Then* h *is a bijection which respects the accessibility relation and the propositional variables and nominals in the common language* $\mathcal{L}'_{\varphi_0\psi_0}$*. Moreover,* $g^{\psi_0} = h \circ g^{\varphi_0}$*, for* g *the natural assignment.*

PROOF OF CLAIM. h is defined at every member of the domain of $\mathfrak{M}_{\varphi_0}$ by Claim 2.*ii* and the fact that for any $n \in$ N, both $\Delta^{\varphi_0}_n$ and $\Delta^{\psi_0}_n$ are uniquely defined. Moreover, h is a bijection because $@_n\top \in T_\omega$ iff $@_n\top \in U_\omega$ and in $\mathfrak{M}_{\varphi_0}$ and $\mathfrak{M}_{\psi_0}$ nominals are interpreted as singletons.

For all proposition variables p in $\mathcal{L}'_{\varphi_0\psi_0}$ we have $\Delta^{\varphi_0}_n \in V^{\varphi_0}(p)$ iff $@_n p \in T_\omega$ iff $@_n p \in U_\omega$ iff $h(\Delta^{\varphi_0}_n) \in V^{\psi_0}(p)$. For the relation R, $(\Delta^{\varphi_0}_n, \Delta^{\varphi_0}_{n'}) \in R^{\varphi_0}$ iff $@_n\Diamond n' \in T_\omega$ iff $@_n\Diamond n' \in U_\omega$ iff $(h(\Delta^{\varphi_0}_n), h(\Delta^{\varphi_0}_{n'})) \in R^{\psi_0}$. A similar argument shows $g^{\psi_0} = h \circ g^{\varphi_0}$. ◀

Since the two models share the same frame, and agree on the common language, there is a model $\mathfrak{M}$ for the union of the two languages which has $\mathfrak{M}_{\varphi_0}$ and $\mathfrak{M}_{\psi_0}$ as reducts. But then by (1), $\mathfrak{M}, g, \Delta_{n_0} \Vdash \varphi_0 \wedge \neg\psi_0$, and we have proved the theorem. QED

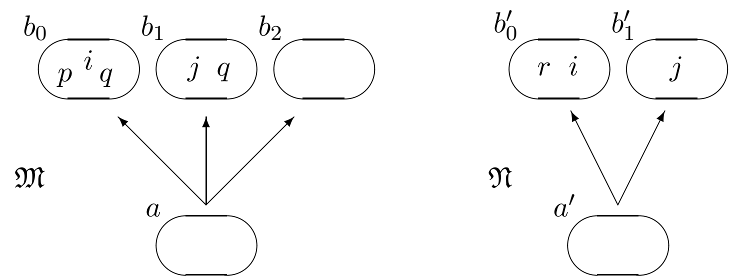
Actually, we can prove a stronger result: we can restrict the free variables occurring in the interpolant θ to only those appearing both in φ_0 and ψ_0, by an easy argument. Moreover, nothing in the proof is intrinsically tied to the number of modalities in the language, i.e., arrow interpolation also holds for the multi-modal versions of $\mathcal{H}(\downarrow, @)$ if modalities are allowed freely in the interpolant. We conjecture that interpolation goes through even if the interpolant's modalities are restricted to the common language.

But the most important generalization is that strong interpolation holds not only in the minimal logic of $\mathcal{H}(\downarrow, @)$ *but in any pure axiomatic extension*. As is shown in [PT85, GG93, BT98a], named models validate pure axioms. Now, we showed how to use named models to prove interpolation in Theorem 4.4. So if we use the same construction for any extension of $\mathcal{H}(\downarrow, @)$ obtained by adding *pure* axioms, the resulting frame will validate the extra axioms. Hence in view of our earlier characterization of the bounded fragment we have:

Theorem 4.5 *Let* $\varphi(\bar{x})$ *be any frame condition in the bounded fragment. The theory in the hybrid language* $\mathcal{H}(\downarrow, @)$ *of the class* FRAMES$(\forall\bar{x}.\varphi(\bar{x}))$ *enjoys strong interpolation (AIP).*

This result stands in sharp contrast to the scarcity of general interpolation results obtained for the basic modal language; see for example [Mak91]. Indeed, it can be viewed as delineating the syntactic form of interpolants in modal logic as follows. Let L be the modal logic in the basic modal language of a first-order definable class FRAMES of frames. Now, even if we cannot find interpolants in the modal language itself, we can always find first-order interpolants by the interpolation theorem for first-order logic and the standard translation. But the last theorem tells us that we don't always have to move to the full first-order language to repair modal interpolation failures: if FRAMES is defined by the universal closure of a theory in the bounded fragment, then we find the interpolants in the bounded fragment.

It is clear from the proof of the interpolation theorem that the number of world variables needed cannot be bounded (they are used to quantify away the nominals in the proof of Claim 1). Indeed, if we restrict $\mathcal{H}(\downarrow, @)$ to only a finite number of variables, then arrow interpolation *fails*. Because we have the notion of a k-bisimulations at our disposal, it is fairly straightforward to provide counterexamples. Let's consider first the case of $\mathcal{H}(\downarrow, @)$ restricted to only one world variable. Take the models



and the formulas

$$\begin{aligned} \varphi &= \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(\neg p \wedge \neg q) \\ \psi &= (\Diamond r \wedge \Box(r \to i)) \to (\Diamond(\neg r \wedge j) \to \Diamond(\neg r \wedge \neg j)). \end{aligned} \tag{2}$$

It is easy to prove that $\varphi \to \psi$ is valid: in any model having a world with at least three different accessible worlds, if there is a unique accessible r-world and one of the accessible $\neg r$-worlds is named by the nominal j, then the second accessible $\neg r$-world is named $\neg j$. Moreover, it is not difficult to see that

$$\theta = \downarrow x.\Diamond(\downarrow y.@_x(\Diamond(\downarrow y'.@_x(\Diamond(y \wedge \neg y') \wedge \Diamond(\neg y \wedge y') \wedge \Diamond(\neg y \wedge \neg y')))))$$

is an (arrow and turnstile) interpolant. But this is a sentence in *three* variables; is there an interpolant containing only *one* world variable?

No. Note that $\mathfrak{M}$ and $\mathfrak{N}$ 1-bisimulate in the common (empty) language via the relation

$$(m, m') \overset{1}{\sim} (n, n') \text{ iff } \text{depth}(m) = \text{depth}(n) \;\&\; \text{depth}(m') = \text{depth}(n') \;\&\; (m = m' \Leftrightarrow n = n'),$$
where depth is the distance from the root.

Furthermore, φ is true in $\mathfrak{M}, a$, while ψ is false at $\mathfrak{N}, a'$ which proves that an interpolant on only one variable does not exist. Indeed, no interpolant on two variables exists either, as a 2-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ can also be defined. Incidentally, while the formulas $\varphi = \Diamond p \wedge \Diamond\neg p$ and $\psi = \Diamond i \to \Diamond\neg i$ provide a simpler counterexample to strong interpolation in the one-variable fragment, they have $\Diamond\top \wedge (\Diamond x \to \Diamond\neg x)$ as weak interpolant; we will use the above example to prove failure in the weak case also.

Notice that the heart of the counterexample is just a counting argument, which can be reproduced for any finite variable fragment of $\mathcal{H}(\downarrow, @)$ by taking bigger and bigger models $\mathfrak{M}$ and $\mathfrak{N}$ exhibiting the same basic pattern. Hence:

Theorem 4.6 *Strong interpolation (AIP) fails in all finite variable fragments of* $\mathcal{H}(\downarrow, @)$.

A more complex counterexample based on the same idea can be set up to prove failure of weak (turnstile) interpolation. Consider again the formulas φ and ψ in (2) above. Clearly $\varphi \models^{glo} \psi$. Take now the model $\mathfrak{M}$ and define $\mathfrak{M}'$ by linking new copies of b_0, b_1 and b_2 to each terminal world in $\mathfrak{M}$. Let $\mathfrak{M}_\omega$ be the infinite model obtained by iterating this operation ω times and similarly for $\mathfrak{N}_\omega$. Now, $\mathfrak{M}_\omega$ makes φ globally true. Suppose θ is an interpolant on one variable. Then as $\varphi \models^{glo} \theta$, θ is globally true at $\mathfrak{M}_\omega$.

We need something stronger than a mere 1-bisimulations linking $\mathfrak{M}_\omega$ and $\mathfrak{N}_\omega$, as we want to transfer *global* truth. With ordinary modal languages, requiring $\sim$ to be total and surjective is enough, but we have to take care of assignments as well. We shall say that a k-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ is *full* if for every pair $\langle \bar{m}, m\rangle \in {}^kM \times M$ there exists $\langle \bar{n}, n\rangle \in {}^kN \times N$ such that $\langle \bar{m}, m\rangle \overset{k}{\sim} \langle \bar{n}, n\rangle$ and vice versa. If we can define a full 1-bisimulation between $\mathfrak{M}_\omega$ and $\mathfrak{N}_\omega$ then $\mathfrak{N}_\omega \models^{glo} \theta$. But $\overset{1}{\sim}$ defined as in the previous case is indeed full. Hence, as $\theta \models^{glo} \psi$, ψ should be globally true in $\mathfrak{N}_\omega$ — but it is not.

Theorem 4.7 *Weak interpolation (TIP) fails in all finite variable fragments of* $\mathcal{H}(\downarrow, @)$.

Finally, we see from the proof of Theorem 4.4 that the $\downarrow$ binder is needed in Claim 1. So what about interpolation in the sublanguage $\mathcal{H}(@)$? We can again use models $\mathfrak{M}$ and $\mathfrak{N}$ to prove that arrow interpolation fails; we use the restricted version of k-bisimulation which leaves out condition ($\downarrow$). In this framework we can define for any k, a k-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ such that for any $\bar{m} \in {}^kM$ and any $\bar{n} \in {}^kN$, $(\bar{m}, a) \overset{k}{\sim} (\bar{n}, a')$. This proves that there is no arrow interpolant for $\varphi \to \psi$ in $\mathcal{H}(@)$ (even when free variables are allowed).

Theorem 4.8 *Strong interpolation (AIP) fails in* $\mathcal{H}(@)$.

But *weak* interpolation holds for $\mathcal{H}(@)$ because the role of $\downarrow$ is played by the implicit quantification in the definition of $\varphi \models^{glo} \psi$. (Another way of looking at it is to say that the definition of $\models^{glo}$ sneaks in implicit quantification using the global hybrid binder $\forall$ mentioned in the introduction.)

Theorem 4.9 *Let φ and ψ be sentences of $\mathcal{H}(@)$ such that $\varphi \models^{glo} \psi$. Then there is a formula θ, which may contain additional free variables, such that*

i. $\varphi \models^{glo} \theta$ *and* $\theta \models^{glo} \psi$.

ii. $I\!\!P(\theta) \subseteq I\!\!P(\varphi) \cap I\!\!P(\psi)$.

OUTLINE OF PROOF. We outline how the proof of arrow interpolation for $\mathcal{H}(\downarrow, @)$ should be modified to obtain the result.

First, the construction of the pasted sets T_ω and U_ω needs to be adjusted as we need to ensure that the labeled models obtained from them make φ_0 and $\neg\psi_0$ *globally* true. To that end, whenever we run into a formula of the form $@_s s$ or $@_s\Diamond\xi$ we paste not only a new nominal n_k but also the formulas we want to make globally true. For example one clause in the definition of T_{j+1} would read

- if $\varphi_j = @_s s$, then $T_{j+1} = T_j \cup \{\varphi_j\} \cup \{@_s(n_k \wedge s \wedge \varphi_0)\}$, for $n_k \in \mathsf{N}\backslash\mathsf{NOM}(T_j \cup U_j)$.

We will need to show that for all $j \in \omega$, $\langle T_j, U_j\rangle$ is (globally) inseparable with respect to $\mathcal{L}'_{\varphi_0\psi_0}$. The base case is simple: if θ (including perhaps some new nominals $\{n_{i_i}, \ldots, n_{i_k}\}$) separates $\langle T_0, U_0\rangle$ on

$\mathcal{L}'_{\varphi_0\psi_0}$, then $\theta[x_{i_1} \dots x_{i_k}]$ separates $\langle\{\varphi_0\}, \{\neg\psi_0\}\rangle$, for new variables $\{x_{i_i}, \dots, x_{i_k}\}$; this is precisely where the free variables in the interpolant are needed.

What about the inductive step? Consider, for example, the case of $\varphi_j = @_s s$. Assume that $\langle T_j \cup \{\varphi_j\}, U_j\rangle$ is inseparable in $\mathcal{L}'_{\varphi_0\psi_0}$; we want to prove that $\langle T_j \cup \{\varphi_j, @_s(n_k \wedge s \wedge \varphi_0)\}, U_j\rangle$ is inseparable. Suppose θ separates this last pair. Then $U_j \models^{glo} \neg\theta$ while $T_j \cup \{@_s s, @_s(n_k \wedge s \wedge \varphi_0)\} \models^{glo} \theta$. Because $@_s(n_k \wedge s \wedge \varphi_0)$ is an @-formula, this is the case iff $T_j \cup \{@_s s\} \models^{glo} @_s(n_k \wedge s \wedge \varphi_0) \rightarrow \theta$. Furthermore, as $\varphi_0 \in T_j$ and n_k is a new nominal by definition, for all $\mathfrak{M}$, $\mathfrak{M} \models T_j$ implies $\mathfrak{M} \models @_s(n_k \wedge s \wedge \varphi_0)$. Hence $T_j \cup \{@_s s\} \models^{glo} \theta$. Contradiction.

From now on the proof follows the same lines as before. We obtain labeled models such that $\mathfrak{M}_{\varphi_0} \models \varphi_0$ and $\mathfrak{M}_{\psi_0} \models \neg\psi_0$ "sharing" the same frame, from which we build a model $\mathfrak{M}$ where $\varphi_0 \wedge \neg\psi_0$ holds globally. QED

It is known that $\mathcal{H}(@)$ is decidable and that $\mathcal{H}(\downarrow, @)$ is undecidable (see the following section). Thus we have a decidable system with TIP, an undecidable system with AIP, and it is natural to ask:

Open Question 4.10 *Is there any decidable hybrid language extending $\mathcal{H}(@)$ that enjoys arrow interpolation?*

To close this section, some remarks on Beth Definability. The Beth Definability Property [Bet53] is commonly studied together with interpolation (indeed, sometimes interpolation is considered to be just a step in the proof of the Beth Property). Loosely speaking, a logic has the Beth property if any implicit definition has also an explicit definition. More precisely, for hybrid languages we define:

Definition 4.11 (Beth Definability) *A hybrid logic has the* Beth Definability Property *if for all formulas $\varphi(\bar{a}, a)$ whose propositional letters, free world variables and nominals occur among $\bar{a}, a$, if $\models \varphi(\bar{a}, a)[a/b_1] \wedge \varphi(\bar{a}, a)[a/b_2] \rightarrow (b_1 \leftrightarrow b_2)$ then there is a formula $\psi(\bar{a})$ such that $\models \varphi(\bar{a}, a) \rightarrow (\psi(\bar{a}) \leftrightarrow a)$.*

Using a standard argument [Cra57] we can easily derive the Beth definability property for $\mathcal{H}(\downarrow, @)$ from Theorem 4.4.

Theorem 4.12 *$\mathcal{H}(\downarrow, @)$ has the Beth definability property.*

A careful analysis of weaker versions of the Beth property can be carried out for fragments of $\mathcal{H}(\downarrow, @)$ as we did with the interpolation property.

5 Complexity

$\mathcal{H}(\downarrow, @)$ is undecidable (this is known from [BS95], unpublished work by Valentin Goranko, and can be proved directly for the bounded fragment [Woo81]), but the sublanguage $\mathcal{H}(@)$ is decidable (this is an easy consequence of results in [PT91, GG93, Bla93]). In this section we examine the computational complexity of $\mathcal{H}(@)$ and related systems, and sharpen the undecidability result for $\mathcal{H}(\downarrow, @)$.

We study (*local*) K-satisfiability problems: given a formula φ, does there exist a model $\mathfrak{M}$, an assignment g, and a world m, such that $\mathfrak{M}, g, m \Vdash \varphi$? (The K reflects the fact that we place no restrictions on the satisfying models: in effect we are measuring the complexity of the minimal logic K in whatever language we are working with.) Note that we don't need to bother about variable assignments when working in $\mathcal{H}(@)$: if we replace all world variables in φ by nominals, obtaining φ', then φ is satisfiable if and only if φ' is, so we can restrict our attention to variable-free $\mathcal{H}(@)$ formulas. For definitions of the complexity classes PSPACE and EXPTIME and other background information, see [Pap94].

5.1 The language $\mathcal{H}(@)$

$\mathcal{H}(@)$ is a well-behaved sublanguage of $\mathcal{H}(\downarrow, @)$. As we have just seen, although $\mathcal{H}(@)$ does not enjoy strong interpolation, it does have weak interpolation. Moreover, simple tableaux and sequent systems for $\mathcal{H}(@)$ can be defined by exploiting the interplay between nominals and @; see [Sel91, Bla00]. Furthermore, while $\mathcal{H}(@)$ doesn't offer any exciting new expressivity at the level of *models* (for example, without the $\downarrow$ binder we can't define *Until*) it does provide new expressivity at the level of *frames*: we can define many properties that are not definable in ordinary propositional modal logic, including irreflexivity ($@_i\Box\neg i$), asymmetry ($@_i\Diamond j \to @_j\neg\Diamond i$) and antisymmetry ($@_i\Diamond j \wedge @_j\Diamond i \to @_i j$); these correspondences are easy to check using the standard translation. Moreover, pure formulas such as these automatically yield complete axiomatizations for the frame classes they define; see [BT98a, BT99, Bla00].

Thus there are many reasons for being interested in $\mathcal{H}(@)$, and a natural question to ask is: how high a computational price do we pay for these benefits? It turns out that (up to a polynomial) there are no extra computational costs when expanding unimodal logic (or even multimodal logic) with $@$ and nominals and/or free variables.

Theorem 5.1 *The* K*-satisfiability problem for a multimodal language enriched with nominals and* $@$ *is* PSPACE*-complete.*

The lower bound follows directly from the PSPACE-hardness of the local K-satisfiability problem in the ordinary unimodal language [Lad77], while a matching upper bound is easy to obtain by means of model-construction games [ABM99].

5.2 The language $\mathcal{H}_t(@)$

Matters are different if we change the underlying modal logic to tense logic. We know from [Spa93b] that the K-satisfiability problem for the usual language of tense logic is PSPACE-complete. However expanding such a language with even a single nominal (or free variable) results in an EXPTIME-hard satisfiability problem. This happens even if we don't add @ — all that's needed is one nominal. We prove this using the *spypoint* technique introduced in [BS95]. We will use a more sophisticated version of this technique later in the paper to sharpen the undecidability result for $\mathcal{H}(\downarrow, @)$.

Theorem 5.2 *The* K*-satisfiability problem for a language of tense logic containing at least one nominal is* EXPTIME*-hard.*

PROOF. We shall reduce the EXPTIME-complete global K-satisfiability problem for ordinary unimodal languages (see [HM92, Spa93a]) to the (local) K-satisfiability problem for a language of tense logic that contains at least one nominal. The global K-satisfiability problem for a modal language is the following: given a formula φ in the modal language, does there exist a model $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi$ (in other words, where φ is true in *all* worlds)?

We use a spypoint argument. Define the following translation function $(\cdot)^t$ from ordinary unimodal formulas to formulas in a tense language that contains at least one nominal i: $p^t = p$, $(\neg\varphi)^t = \neg\varphi^t$, $(\varphi \wedge \psi)^t = \varphi^t \wedge \psi^t$, $(\Diamond\varphi)^t = \mathsf{F}(\mathsf{P}i \wedge \varphi^t)$; i is a *fixed* nominal in this translation. Clearly $(\cdot)^t$ is a linear reduction. We claim that for any formula φ, φ is globally K-satisfiable if and only if $i \wedge \mathsf{F}\neg i \wedge \mathsf{G}(\mathsf{P}i \to \varphi^t)$ is (locally) K-satisfiable.

For the left to right direction, let $\mathfrak{M} \models \varphi$, where $\mathfrak{M} = \langle M, R, V\rangle$ is an ordinary Kripke model. Define $\mathfrak{M}^*$ as follows: $M^* = M \cup \{i\}$, $R^* = R \cup \{(i, m) \mid m \in M\}$, and $V^* = V \cup \{(n, \{i\}) \mid$

for all nominals $n\}$. $\mathfrak{M}^*$ is a hybrid model, for all nominals (including i) are interpreted by the singleton set $\{i\}$, our spypoint. We claim that for all $m \in M$, for all ψ, we have $\mathfrak{M}, m \Vdash \psi$ if and only if $\mathfrak{M}^*, m \Vdash \psi^t$. This follows by induction. The interesting step is for $\Diamond$:

$$
\begin{array}{ll}
& \mathfrak{M}, m \Vdash \Diamond\psi \\
\Leftrightarrow & (\exists m' \in M) : Rmm' \;\&\; \mathfrak{M}, m' \Vdash \psi \\
\Leftrightarrow & (\exists m' \in M^*) : R^*mm' \;\&\; \mathfrak{M}^*, m' \Vdash \psi^t \;\&\; R^*im' \text{ (by IH and definition of } R^*) \\
\Leftrightarrow & \mathfrak{M}^*, m \Vdash \mathsf{F}(\mathsf{P}i \wedge \psi^t) \\
\Leftrightarrow & \mathfrak{M}^*, m \Vdash (\Diamond\psi)^t.
\end{array}
$$

It follows that $\mathfrak{M}^*, i \Vdash i \wedge \mathsf{F}\neg i \wedge \mathsf{G}(\mathsf{P}i \rightarrow \varphi^t)$, as desired.

For the other direction, let $\mathfrak{M}, w \Vdash i \wedge \mathsf{F}\neg i \wedge \mathsf{G}(\mathsf{P}i \rightarrow \varphi^t)$, where $\mathfrak{M} = \langle M, R, V\rangle$ is a hybrid model. Define $\mathfrak{M}^*$ as follows: $M^* = \{m \in M \mid Rwm\}$, $R^* = R_{\restriction M^*}$, $V^* = V_{\restriction M^*}$. Note that M^* is not empty, for $\mathfrak{M}, w \Vdash \mathsf{F}\neg i$. We claim that for all $m \in M^*$, for all ψ, $\mathfrak{M}, m \Vdash \psi^t$ if and only if $\mathfrak{M}^*, m \Vdash \psi$. Again we only present the inductive step for $\Diamond$:

$$
\begin{array}{ll}
& \mathfrak{M}, m \Vdash \mathsf{F}(\mathsf{P}i \wedge \psi^t) \\
\Leftrightarrow & (\exists m' \in M) : Rmm' \;\&\; Rwm' \;\&\; \mathfrak{M}, m' \Vdash \psi^t \\
\Leftrightarrow & (\exists m' \in M^*) : Rmm' \;\&\; Rwm' \;\&\; \mathfrak{M}, m' \Vdash \psi^t \\
\Leftrightarrow & (\exists m' \in M^*) : R^*mm' \;\&\; \mathfrak{M}^*, m' \Vdash \psi \text{ (by IH and definition of } M^*) \\
\Leftrightarrow & \mathfrak{M}^*, m \Vdash \Diamond\psi.
\end{array}
$$

For all $m \in M^*$, Rwm holds, whence for all $m \in M^*$, $\mathfrak{M}, m \Vdash \mathsf{P}i$. So, since $\mathfrak{M}, w \Vdash \mathsf{G}(\mathsf{P}i \rightarrow \varphi^t)$, for all $m \in M^*$, $\mathfrak{M}, m \Vdash \varphi^t$. Hence by our last claim $\mathfrak{M}^* \models \varphi$. QED

What about a matching upper bound? In fact, even though the addition of just one nominal to the language of tense logic yields an EXPTIME-hard K-satisfiability problem, adding further nominals, multiple forward and backward looking modalities, and the universal modality A too, does *not* take us any higher in the complexity hierarchy.

This can be established by extending known results for nominal Propositional Dynamic Logic. The satisfiability problem for Propositional Dynamic Logic (PDL) enriched with both nominals and A is solvable in EXPTIME (see [PT91]). Moreover, as De Giacomo observes in [De 95], his results on PDL-like description languages containing the $\mathcal{O}$ ("one-of") operator show that the satisfiability problem for nominal PDL *with converse programs* is solvable in EXPTIME too. Now, on *connected* frames — assuming a finite repertoire of atomic programs — the universal modality is definable in converse PDL. But to establish the upper bounds we want, we need to know that we can have access to both converse programs and A on *arbitrary* frames and still stay in EXPTIME. And in fact, we can:

Theorem 5.3 *The satisfiability problem for nominal Propositional Dynamic Logic with converse programs and the universal modality is solvable in* EXPTIME.

PROOF. Proved in [ABM00] using a spypoint argument. QED

Corollary 5.4 *The local* K*-satisfiability problem for $\mathcal{H}_t(@)$ plus the universal modality is* EXPTIME-*complete.*

Because of the availability of the Kleene star in PDL, we can also establish the EXPTIME-completeness of $\mathcal{H}_t(@)$ over transitive frames. More interestingly, the complexity of $\mathcal{H}_t(@)$ drops back to PSPACE or below when considering more structured classes of frames, such as linear orders or transitive trees. For a full discussion, see [ABM00].

5.3 Sharpening Undecidability

When $\mathcal{H}(\downarrow, @)$ is first encountered, a common reaction is that it must be decidable: it seems plausible that some sort of "$\downarrow$-elimination" argument could reduce its satisfiability problem to that of $\mathcal{H}(@)$. But Theorem 3.11 tells us that every formula in the bounded fragment of first-order logic is equivalent to an $\mathcal{H}(\downarrow, @)$ formula, so it should be clear that this cannot be done. In fact, $\mathcal{H}(\downarrow, @)$ is undecidable, and by using a more sophisticated spypoint argument we can show something even stronger:

> *The fragment of* $\mathcal{H}(\downarrow)$ *consisting of* pure nominal-free sentences *has an undecidable satisfiability problem.*

We proceed as follows. We first sketch an easy undecidability proof for the full language $\mathcal{H}(\downarrow, @)$. By generalizing the underlying argument, we will be lead to the *Spypoint Theorem* and the undecidability result just stated.

In [Spa93b] it is shown that the ordinary modal global satisfiability problem for the class K_{23} (that is, the class of frames $\langle W, R\rangle$ in which every state has at most 2 R-successors and at most 3 two-step R-successors) is undecidable. We shall reduce this problem to the satisfiability problem for $\mathcal{H}(\downarrow, @)$.

Let $Grid$ be the conjunction of the following formulas:

$$
\begin{array}{ll}
G_1 & @_s\neg\Diamond s\\
G_2 & @_s\Diamond\top\\
G_3 & @_s(\Box\Box\downarrow x.@_s\Diamond x)\\
G_4 & @_s(\Box\downarrow y.\Box\downarrow x_1.@_y\Box\downarrow x_2.@_y\Box\downarrow x_3.(@_{x_1}x_2 \vee @_{x_1}x_3 \vee @_{x_2}x_3))\\
G_5 & @_s(\Box\downarrow y.\Box\Box\downarrow x_1.@_y\Box\Box\downarrow x_2.@_y\Box\Box\downarrow x_3.@_y\Box\Box\downarrow x_4.(\bigvee_{1\leq i\neq j\leq 4} @_{x_i}x_j)).
\end{array}
$$

What does $Grid$ express? Suppose it is satisfied in a model $\mathfrak{M}$ on a frame $\langle M, R\rangle$. Then there exists a state which is named by s (the *spypoint*). By G_1, s is not related to itself. By G_2, s is related to some state, and by G_3, every state which can be reached from s in two steps can also be reached from s in one step. This means that in $\mathfrak{M}_s$ — the submodel of $\mathfrak{M}$ generated by s — every state is reachable from s in one step. Now, G_4 and G_5 express precisely the two conditions characterizing the class K_{23} on successors of s. To get the intuition, note that the simple formula $@_s\Box\downarrow y.\Box\downarrow x_1.@_y\Box\downarrow x_2.@_{x_1}x_2$ expresses that every successor of s in $\mathfrak{M}_s$ has at most one R-successor. G_4 and G_5 follow the same pattern.

We claim that for every formula φ,

$$\varphi \text{ is globally satisfiable on a } \mathsf{K}_{23}\text{-frame iff } Grid \wedge @_s\Box\varphi \text{ is satisfiable.}$$

The proof is a simple copy of the two constructions given in the proof of Theorem 5.2.

We shall now sharpen this result. We do so by analyzing the undecidability proof just given and generalizing the underlying ideas. The model we used had a certain characteristic form. Let's pin this down:

Definition 5.5 *A model* $\mathfrak{M} = \langle M, R, V\rangle$ *is called a* spypoint model *if there is an element* $s \in M$ *(the spypoint) such that*

- *i.* $\neg sRs$*;*
- *ii. For all* $w \in M$*, if* $w \neq s$*, then* sRw *and* wRs*.*

Notice that by *ii.* above, any spypoint model is generated by its spy point. We will now show that with $\downarrow$ we can easily create spypoint models. On these models we can create for every variable x introduced by $\downarrow x$, a formula which has precisely the meaning of $@_x$.

Proposition 5.6 *Let $\mathfrak{M} = \langle M, R, V\rangle$ and $s \in M$ be such that $\mathfrak{M}, s \Vdash \downarrow s.(\neg\Diamond s \wedge \Box\Box \downarrow x.\Diamond(s \wedge \Diamond x) \wedge \Box\Diamond s)$. Then,*

i. $\mathfrak{M}_s$, the submodel of $\mathfrak{M}$ generated by s, is a spypoint model with s the spypoint.

ii. $@_s\varphi$ is definable on $\mathfrak{M}_s$ by $(s \wedge \varphi) \vee \Diamond(s \wedge \varphi)$.

iii. Let g be any assignment. Then for all $u \in M$, $\mathfrak{M}_s, g, u \Vdash @_x\varphi$ iff $\mathfrak{M}_s, g, u \Vdash @_s(\varphi \vee \Diamond(x \wedge \varphi))$.

PROOF. *i.* is immediate. *ii.* and *iii.* follow from the properties of spypoint models. QED

Now, spypoint models are very powerful: we can encode lots of information about Kripke models (for finitely many propositional variables) inside a spypoint model. More precisely, for each Kripke model $\mathfrak{M}$, we define the notion of *a spypoint model of* $\mathfrak{M}$.

Definition 5.7 *Let $\mathfrak{M} = \langle M, R, V\rangle$ be a Kripke model in which the domain of V is a finite set $\{p_1, \ldots, p_n\}$ of propositional variables. The spypoint model of $\mathfrak{M}$ (notation $\mathsf{Spy}[\mathfrak{M}]$) is the structure $\langle M', R', V'\rangle$ in which*

i. $M' = M \cup \{s\} \cup \{w_{p_1}, \ldots, w_{p_n}\}$, for $s, w_{p_1}, \ldots, w_{p_n} \notin M$

ii. $R' = R \cup \{(s, x), (x, s) \mid x \in M'\backslash\{s\}\} \cup \{(x, w_{p_i}) \mid x \in M$ and $x \in V(p_i)\}$

iii. $V' = \emptyset$.

Let $\{s, x_{p_1}, \ldots, x_{p_n}\}$ be a set of state variables. A spypoint assignment *for this set is an assignment g which sends s to the spypoint s and x_{p_i} to w_{p_i}. We use $\mathbf{m}$ as an abbreviation for $\neg s \wedge \neg x_{p_1} \wedge \ldots \wedge \neg x_{p_n}$. Note that when evaluated under a spypoint assignment, the denotation of $\mathbf{m}$ in $\mathsf{Spy}[\mathfrak{M}]$ is precisely M.*

$\mathsf{Spy}[\mathfrak{M}]$ encodes the valuation on $\mathfrak{M}$ and we can take advantage of this fact. Define the following translation from unimodal formulas in variables $\{p_1, \ldots, p_n\}$ to hybrid formulas:

$$\begin{array}{lcl} IT(p_i) & = & \Diamond(x_{p_i}) \\ IT(\neg\varphi) & = & \neg IT(\varphi) \\ IT(\varphi \wedge \psi) & = & IT(\varphi) \wedge IT(\psi) \\ IT(\Diamond\varphi) & = & \Diamond(\mathbf{m} \wedge IT(\varphi)). \end{array}$$

Proposition 5.8 *Let $\mathfrak{M}$ be a Kripke model and φ a unimodal formula. Then for any spypoint assignment g,*

$$\mathfrak{M} \models \varphi \textit{ if and only if } \mathsf{Spy}[\mathfrak{M}], g, s \Vdash \Box(\mathbf{m} \rightarrow IT(\varphi)).$$

PROOF. Immediate by the fact that the spypoint is R-related to all states in the domain of $\mathfrak{M}$, and the interpretation of $\mathbf{m}$ under any spypoint assignment g. QED

We modify the hybrid translation HT to its relativized version $HT^{\mathbf{m}}$ which also defines away occurrences of $@$. Define $HT^{\mathbf{m}}(\exists v.(Rtv \wedge \varphi))$ as $@_t\Diamond \downarrow v.(\mathbf{m} \wedge HT^{\mathbf{m}}\varphi)$ and replace all $@$ symbols by their definition as indicated in Proposition 5.6.*ii* and 5.6.*iii*.

The crucial step is now the fact that $\downarrow$ is strong enough to encode many frame-conditions.

Proposition 5.9 *Let $\mathfrak{M} = \langle M, R, V\rangle$ be a Kripke model. Let $C(y)$ be a formula in the bounded fragment in the signature $\{R, =\}$. Then for any spypoint assignment g,*

$$\langle M, R\rangle \models \forall y.C(y) \textit{ if and only if } \mathsf{Spy}[\mathfrak{M}], g, s \Vdash \Box \downarrow y.(\mathbf{m} \rightarrow HT^{\mathbf{m}}(C(y))).$$

PROOF. Immediate by the properties of HT, Proposition 5.6, and the fact that the spypoint is R-related to all states in the domain of $\mathfrak{M}$. QED

Theorem 5.10 (Spypoint Theorem) *Let φ be a unimodal formula in $\{p_1, \ldots, p_n\}$ and $\forall y.C(y)$ a first-order frame condition in $\{R, =\}$ with $C(y)$ in the bounded fragment. The following are equivalent.*

i. There exists a Kripke model $\mathfrak{M} = \langle M, R, V\rangle$ such that $\langle M, R\rangle \models \forall y.C(y)$ and $\mathfrak{M} \models \varphi$.

ii. The pure hybrid sentence F in the language $\mathcal{H}(\downarrow)$ is satisfiable. F is

$$\downarrow s.(SPY \wedge \Diamond \downarrow x_{p_1}.@_s\Diamond \downarrow x_{p_2}@_s \ldots \Diamond \downarrow x_{p_n}.@_s(DIS \wedge VAL \wedge FR)),$$

where

$$\begin{array}{lcl} SPY & = & \neg\Diamond s \wedge \Box\Box \downarrow x.\Diamond(s \wedge \Diamond x) \wedge \Box\Diamond s \\ DIS & = & \Box(\bigwedge_{1\leq i\leq n}(x_{p_i} \rightarrow \bigwedge\{\neg x_{p_j} \mid 1 \leq j \neq i \leq n\})) \\ VAL & = & \Box(\mathbf{m} \rightarrow IT(\varphi)) \\ FR & = & \Box \downarrow y.(\mathbf{m} \rightarrow HT^{\mathbf{m}}(C(y))). \end{array}$$

PROOF. The way we have written it, F contains occurrences of $@_s$; but this does not matter, by Proposition 5.6 all these occurrences can be term-defined. So let's check that F works as claimed.

For the implication from *i* to *ii*, let $\mathfrak{M}$ be a Kripke model as in *i*. We claim that $\mathsf{Spy}[\mathfrak{M}], s \Vdash F$. The first conjunct of F is true in $\mathsf{Spy}[\mathfrak{M}]$ at s by Proposition 5.6. The diamond part of the second conjunct can be satisfied using any spypoint assignment g. In the spypoint model all w_{p_i} are pairwise disjoint, whence $\mathsf{Spy}[\mathfrak{M}], g, s \Vdash DIS$. By Propositions 5.8 and 5.9, also $\mathsf{Spy}[\mathfrak{M}], g, s \Vdash VAL \wedge FR$.

For the other direction, let $\mathfrak{M}, s \Vdash F$. By Proposition 5.6, the submodel $\mathfrak{M}_s = \langle M_s, R_s, V_s\rangle$ generated by s is a spypoint model. Let g be the assignment such that $\mathfrak{M}, g, s \Vdash DIS \wedge VAL \wedge FR$. By DIS, $g(x_{p_i}) \neq g(x_{p_j})$ for all $i \neq j$, and (since $\neg sRs$) also $g(x_{p_i}) \neq s$, for all i. Define the following Kripke model $\mathfrak{M}' = \langle M', R', V'\rangle$, where

$$\begin{array}{lcl} M' & = & M \setminus \{g(s), g(x_{p_1}), \ldots, g(x_{p_n})\} \\ R' & = & R \upharpoonright_{M'} \\ V'(p_i) & = & \{w \mid wRg(x_{p_i})\}. \end{array}$$

Note that $\mathsf{Spy}[\mathfrak{M}']$ is precisely $\mathfrak{M}_s$, and g is a spypoint assignment. But then by Propositions 5.8 and 5.9 and the fact that $\mathfrak{M}_s, g, s \Vdash VAL \wedge FR$, we obtain $\mathfrak{M}' \models \varphi$ and $\langle M', R'\rangle \models \forall y.C(y)$. QED

The proof of the claimed undecidability result is now straightforward.

Corollary 5.11 *The fragment of $\mathcal{H}(\downarrow)$ consisting of all pure nominal-free sentences has an undecidable satisfiability problem.*

PROOF. We will reduce the undecidable global satisfiability problem in the unimodal language over the class K_{23}, just as we did in our easy undecidability result for $\mathcal{H}(\downarrow, @)$. The first-order frame conditions defining K_{23} are of the form $\forall y.C(y)$ with $C(y)$ in the bounded fragment. This is easy to check. For instance, y has at most two successors can be written as

$$\forall x_1(yRx_1 \rightarrow \forall x_2(yRx_2 \rightarrow \forall x_3(yRx_3 \rightarrow (x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3)))).$$

Now apply the Spypoint Theorem. The formula F (after all occurrences of $@_s$ have been term-defined) is a pure nominal-free sentence of $\mathcal{H}(\downarrow)$, and the result follows. QED

6 Further Work

In their long (if sparse) history, hybrid languages have attracted a number of enthusiastic advocates. Some have claimed that hybridization is a natural way to "power-up" the expressive power of modal languages, others have been impressed by the proof theoretical options they open up, or the ease with which general completeness results can be proved. And underlying most of this work lies a simple (and seductive) idea: that by exploiting the notion of formulas as terms to the full, it should be possible to define systems which in some sense combine the best of modal and classical techniques.

We believe that our results confirm the interest of hybridization. For a start, the characterization of $\mathcal{H}(\downarrow, @)$ shows that relatively simple tools are capable of capturing first-order fragments that are central from a modal perspective: invariance under generated submodels mirrors the key notion of locality, and it is pleasing that it can be pinned down so simply. Furthermore, the results on interpolation and complexity tend to confirm that we are dealing with a natural collection of ideas, ideas that are well behaved even in relatively weak sublanguages. Finally, in writing this paper it has become very clear to us that working with hybrid languages involves a genuine interplay of modal and classical methods (for example, both Ehrenfeucht-Fraïssé games and bisimulations were involved in the expressivity result, and interpolation was proved by blending modal canonical models with classical Henkin models). This is something previous writers on hybrid languages have emphasized (see for example [PT91]) and the natural way these methods blend bodes well for further developments.

Nonetheless, to close this paper it seems more appropriate to emphasize what remains to be done — for the fact remains that compared with orthodox modal languages, the study of hybrid languages is in its infancy. Many fundamental questions have not been satisfactorily resolved, and to close the paper we are going to discuss one we regard as particularly important:

> *Which classes of frames are definable using $\mathcal{H}(@)$ formulas whose only atoms are world variables (or equivalently: nominals)?*

In a sense, the standard translation ST already gives us an answer. Let F be a class of frames defined by a sentence φ of the first-order frame language. Then F is definable by a formula of $\mathcal{H}(@)$ whose only atoms are world variables iff there is some formula α in this fragment such that φ is equivalent to the universal closure of $ST(\alpha)$.

Unfortunately, this is not very helpful. Ideally we would like a syntactic characterization of the range of ST when restricted to $\mathcal{H}(@)$ formulas whose only atoms are world variables, together with a reverse translation (like our earlier HT). But at present we only have partial results in this direction.

What about a semantic characterization? Here we can do a little better by introducing the concept of an *@-bisimulation*. Let $\mathfrak{M}$ and $\mathfrak{N}$ be models and $\bar{m} \in {}^k M$ and $\bar{n} \in {}^k N$. A relation $B \subseteq M \times N$ is called an *@-k-bisimulation* between $(\mathfrak{M}, \bar{m})$ and $(\mathfrak{N}, \bar{n})$ if it satisfies the following conditions:

i. B is a modal bisimulation;
ii. $(\forall 0 \le i < k) : (m_i, n_i) \in B$;
iii. $(\forall x \in N) : (m_i, x) \in B \Rightarrow x = n_i$;
iv. $(\forall x \in M) : (x, n_i) \in B \Rightarrow x = m_i$.

We denote this by $(\mathfrak{M}, \bar{m}) \overset{B}{\sim} (\mathfrak{N}, \bar{n})$. A first-order formula $\varphi(\bar{x}, y)$ is called invariant for @-k-bisimulation, if for all models $\mathfrak{M}$, $\mathfrak{N}$, for all k-tuples $\bar{m}$ and $\bar{n}$, for all B such that $(\mathfrak{M}, \bar{m}) \overset{B}{\sim} (\mathfrak{N}, \bar{n})$,

$$(m, n) \in B \Rightarrow \mathfrak{M} \models \varphi(\bar{m}, m) \Leftrightarrow \mathfrak{N} \models \varphi(\bar{n}, n).$$

Theorem 6.1 *A first-order formula $\varphi(\bar{x}, y)$ is invariant for @-k-bisimulation if and only if it is equivalent to the standard translation of an $\mathcal{H}(@)$-formula containing the variables $\bar{x}$.*

PROOF. The proof follows the standard modal lines. We only sketch the differences due to the new operator and the variables. For the preservation part, suppose $(\mathfrak{M}, \bar{m}) \overset{B}{\sim} (\mathfrak{N}, \bar{n})$ and $(m, n) \in B$. Then $\mathfrak{M}, \bar{m}, m \Vdash v_i \Leftrightarrow m = m_i$; but this implies that $n = n_i$, whence $\mathfrak{N}, \bar{n}, n \Vdash v_i$. For @-formulas, $\mathfrak{M}, \bar{m}, m \Vdash @_{v_i}\varphi \Leftrightarrow \mathfrak{M}, \bar{m}, m_i \Vdash \varphi$; but then since $(m_i, n_i) \in B$ by the inductive hypothesis we have that $\mathfrak{N}, \bar{n}, n_i \Vdash \varphi \Leftrightarrow \mathfrak{N}, \bar{n}, n \Vdash @_{v_i}\varphi$.

For the characterization part we use the standard diagram-chasing argument. Let $\mathfrak{M}, \bar{m}, m$ and $\mathfrak{N}, \bar{n}, n$ have the same hybrid theory. Define a relation B between elements in the ω-saturated extensions $\mathfrak{M}^+$ and $\mathfrak{N}^+$ as follows:

$$(x, y) \in B \Leftrightarrow \forall \varphi.(\mathfrak{M}^+, \bar{m}, x \Vdash \varphi \Leftrightarrow \mathfrak{N}^+, \bar{n}, y \Vdash \varphi).$$

The standard proof shows that B is a modal bisimulation. We check the extra conditions. For all i, $(m_i, n_i) \in B$ holds by the following argument. $\mathfrak{M}^+, \bar{m}, m_i \Vdash \varphi \Leftrightarrow \mathfrak{M}^+, \bar{m}, m \Vdash @_{v_i}\varphi \Leftrightarrow \mathfrak{N}^+, \bar{n}, n \Vdash @_{v_i}\varphi \Leftrightarrow \mathfrak{N}^+, \bar{n}, n_i \Vdash \varphi$. The other two conditions are satisfied because of the following. Let $(m_i, y) \in B$. Since $\mathfrak{M}^+, \bar{m}, m_i \Vdash v_i$, also $\mathfrak{N}^+, \bar{n}, y \Vdash v_i$. But then $y = n_i$. This finishes the outline of the proof. QED

Using this result it is easy to show, for example, that $\exists y.(Rxy \wedge Ryy)$ is not equivalent to an @-formula with one free variable, and that $Rxy \wedge Rxz \rightarrow \exists w.(Ryw \wedge Rzw)$ is not equivalent to an @-formula in three free variables. And it does tell us something about frame definability:

Corollary 6.2 *Let* F *be a class of frames defined by a sentence* φ *of the first-order frame language. Then* F *is definable by a formula of* $\mathcal{H}(@)$ *whose only atoms are variables iff* φ *is equivalent to the universal closure of a formula that is invariant under @-*k*-bisimulations.*

But in practice this characterization does not seem to be particularly helpful. An important aspect of our characterization of $\mathcal{H}(\downarrow, @)$ was the way the notion of k-bisimulation linked with the notion of generated submodels to yield a natural "geometric" characterization of definable frame classes. It is hard to see what the definition of @-k-bisimulation invariance tells us about frame geometry. Clearly new ideas are called for here.

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