Logics for Computation

Lecture #10: Were do We Go from Here?

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The Story up to Now

- In the last three lectures we have discussed a very strong logic namely first-order logic (developed using the Arthur Prior style notation $\langle x \rangle$ and $[x]$) from the perspective of inference, expressivity, and computation.
- As we have seen, it is deductively natural, highly expressive (albeit with some interesting limitations), and undecidable.
- The question now, of course, is where (if anywhere) do we go from here . . . ?
- The answer is — higher-order logic, and in particular, second order logic.

What's that?

- Well, what is that? Aren't we already quantifying over everything that there is in our models?
- The answer is no. There's lot more sitting out there in our models, patiently waiting to be quantified.
- Sure, we're already quantifying over the individuals — but there are higher-order entities there too, such as sets of individuals, and relations.
- And these logics certainly do offer increased expressivity...

Transitive closure

- In Lecture 6 we met the concept of the reflexive transitive closure of a relation.
- There are two (equivalent) ways of defining reflexive transitive closure.
  - As the smallest reflexive and transitive relation S (on the domain D) containing an arbitrary relation R; or
  - As the relation T on D defined by xTy iff there is a finite sequence of elements of D such that $x = d_0$ and

$$d_0 R' d_1, d_1 R' d_2, \ldots, d_{n1} R'_n, \text{ and } d_n R'_y$$

    where $dR'e$ means that dRe or $d = e$.
- Let's try defining this concept in our shiny new $\langle x \rangle$ $[x]$ language . . .

Let's try. . .

Let X and Y be binary relations

Let's try...

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$$Ref(X) \quad =_{def} \quad [n](n : \langle X \rangle n).$$

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And it's easy to say that X is transitive:

$$Tran(X) \quad =_{def} \quad [n](n : \langle X \rangle \langle X \rangle n \rightarrow \langle X \rangle n)$$

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And to say that X is a subrelation of Y

$$X \subseteq Y \quad =_{def} \quad [n](n : \langle X \rangle n \rightarrow \langle Y \rangle n).$$

Here's a first try...

So let's put all this together to define transitive reflexive closure:

$Tran^*(R, S)$

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$$\begin{aligned} Tran^*(R,S) \quad =_{def} \quad & Ref(S) \\ & \wedge Tran(S) \end{aligned}$$

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& \wedge \mathit{Tran}(S) \\
& \wedge R \subseteq S \\
& \wedge \textit{S-is-the-smallest-such-subrelation-of-R}
\end{aligned}
$$

Here's a first try. . .

So let's put all this together to define transitive reflexive closure:

$$\begin{aligned} \mathit{Tran}^*(R, S) \quad =_{\mathit{def}} \quad & \mathit{Ref}(S) \\ & \land \mathit{Tran}(S) \\ & \land R \subseteq S \\ & \land \mathit{S\text{-}is\text{-}the\text{-}smallest\text{-}such\text{-}subrelation\text{-}of\text{-}R} \end{aligned}$$

Oh dear. . . !

No way José!

- Try as you might, you won't be able to do this
- And we can prove this using the Compactness Theorem
- $\{\neg p, [R]\neg p, [R][R]\neg p, [R][R][R]\neg p, \ldots, \langle R^* \rangle p\}$
- Every finite sunset has a model. Hence (by Compactness) so does the whole thing. But this is impossible.
- Hence we can define R^*.

So extend the language

- As we learned in Lecture 6, we're free to extend the language.
- Now of course, we could just add the $\langle R^* \rangle$ operator — but that was just one example of something we couldn't do.
- Lets give ourselves the power to quantify over two types of higher order entities: properties and binary relations.

A second order language

- $\langle p \rangle \varphi$, and $[p]\varphi$ express existential and universal quantification over properties.
- $\langle R \rangle \varphi$, and $[R]\varphi$ express existential and universal quantification over relations.
- Semantics? Simply extend what we did in first-order case.

Now we can define reflexive transitive closure. . .

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\mathit{Tran}^*(R,S) \quad =_{def} \quad & \mathit{Ref}(S) \\
& \wedge \mathit{Tran}(S) \\
& \wedge R \subseteq S \\
& \wedge [X](\mathit{Ref}(X) \wedge \mathit{Tran}(X) \wedge R \subseteq X \rightarrow S \subseteq X).
\end{aligned}
$$

What's the Price

- Loss of Completeness (for standard models)
- Loss of Compactness. After all:

  $\{\neg p, [R]\neg p, [R][R]\neg p, [R][R][R]\neg p, \ldots, \langle R^* \rangle p\}$

  is now an example of a set in which each finite subset has a model, and the complete set doesn't.
- Loss of Löwenheim Skolem. (It is easy to define the natural number **N** and the integers **Z** up to isomorphism.)

Tradeoff: expressivity versus computation and inference

- Which brings us back to the fundamental trade-off, expressivity versus inference/tractability.
- We've bought serious expressivity — and have lost everything else.

What we covered in the course

- We've been essentially looking at a menu of logics.
- But the menu was designed by a Master Chef (Tarski!); the meal is built around the crucial ingredient of relational structures.
- Relational structures tell us why logic is applicable in semantics (natural language metaphysics) and computer science.
- Back to a logicist position, but not in traditional sense.
- Monotheist — but not in terms of logic, rather, in terms of semantics.

Relevant Bibliography

And, hanging over it all, the brooding specter of Rudolf Carnap and Hans Reichenbach, the Vienna Circle of Philosophy and the rise of symbolic logic. A muddy world, in which he did not care to involve himself. From: Galactic Pot-Healer, by Philip K. Dick, 1969.