Logics for Computation

Lecture #7: DIY First-Order Logic

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The Story so Far

- Yesterday we saw how we can introduce different operators...
- ...and 'cook' our own logic.
- Now, Patrick wants to talk about First Order Logic after the pause today.
- And he asked whether we can do something about it.
- What do you think? Can we mix the First Order Recipe?

What do we do Today

- We'll cook First Order Logic à la ESSLLI08.
- We will see what we can reuse of what we already have...
- ...and extend the language if necessary.
- We will then show that the language we obtain is actually equivalent to the 'classical' First Order Language.

Checking our Stock

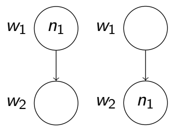
- We want to define for Patrick, a language equivalent to First Order Logic
  (with equality and constants, but without function symbols, he told me he doesn't need them).
- During the previous lectures we introduced a number of operators.
  - Propositional symbols $\surd$
  - The boolean operators $\wedge, \neg$ $\surd$
  - The $\langle R \rangle$ operator $\surd$
  - Constants $\surd$
  - The : operator $\surd$
  - The counting operators $\langle = n\ R \rangle$ $\times$
  - The universal operator $[U]$ Close, but no cigar!
  - The reflexive and transitive closure operator $\langle R^* \rangle$ $\times$
- Which one can we use?

The $[U]$ operator is not enough

- Granted: we need universal quantification.
- But the $[U]$ operator is not expressive enough.
  - We won't prove it here (one way to do it, for example is noting that the language containing $[U]$ is still decidable, while full first order logic should be undecidable).
  - The universal operator is not fine grained enough:
    $[U]$ says for all
    and we need for all x
- First order quantification gives as a delicate control (via variables) of what we are quantifying on.

A Detour: Renaming Points

- I will define a litle piece of notation that I will need in the next slide.
- As I want it to be very clear, I'll do it here and give an example.
- Let
  - $\mathcal{M} = \langle D, \{R_i\}, \{P_i\}, \{N_i\} \rangle$ be a model,
  - w an element in D ($w \in D$),
  - and n_i a name.
- We write $\mathcal{M}[n_i{:=}w]$ for the model obtained from $\mathcal{M}$ where the only change is that now n_i is interpreted as w.



$\mathcal{M} \Rightarrow \mathcal{M}[n_1{:=}w_2]$

First Order Quantification

- We introduce the operator $\langle n \rangle$ where n is a name (we will call the operator rename n) as:
  $\mathcal{M}, w \models \langle n \rangle \varphi$ iff for some w' $\mathcal{M}[n{:=}w'], w \models \varphi$
- Compare with
  $\mathcal{M}, w \models \langle R \rangle \varphi$ iff there is w' s.t. wRw' and $\mathcal{M}, w' \models \varphi$.
- Compare with $\langle U \rangle \varphi := \neg [U] \neg \varphi$
  $\mathcal{M}, w \models \langle U \rangle \varphi$ iff for sum $w', \mathcal{M}, w' \models \varphi$
- Actually, using $\langle n \rangle$ and : together we can define $[U]$:

$$[U]\varphi \text{ iff } \neg \langle n \rangle (n{:}\neg\varphi)$$

Capturing all First Order Logic

- We will now show that we can capture all First Order Logic using: the $\langle R \rangle$ language, names, : and $\langle n \rangle$.
- We will (recursively) define a translation that will assign to each formula of the First Order Language, and equivalent formula in our language

$$\begin{array}{rcl}
Tr(s = t) & = & s{:}t \\
Tr(P(s)) & = & s{:}p \\
Tr(R(s,t)) & = & s{:}\langle R \rangle t \\
Tr(\neg\varphi) & = & \neg Tr(\varphi) \\
Tr(\varphi \wedge \psi) & = & Tr(\varphi) \wedge Tr(\psi) \\
Tr(\exists s.\varphi) & = & \langle s \rangle Tr(\varphi) \\
(\quad Tr(\forall s.\varphi) & = & \neg \langle s \rangle \neg Tr(\varphi) = [x] Tr(\varphi) \quad)
\end{array}$$

Examples

- Let's write down a couple of formulas in First Order Logic and translate them to our language. Again, let's use the convention $[X]$ for $\neg\langle X\rangle\neg$

$$
\begin{array}{c}
Tr(\forall x.(Man(x) \rightarrow \exists y.(Woman(y) \wedge Loves(x,y)))\ ) \\
[x](Tr(Man(x) \rightarrow \exists y.(Woman(y) \wedge Loves(x,y)))) \\
[x](\ Tr(Man(x)) \rightarrow Tr(\exists y.(Woman(y) \wedge Loves(x,y)))) \\
[x](x{:}Man \rightarrow Tr(\exists y.(Woman(y) \wedge Loves(x,y)))) \\
[x](x{:}Man \rightarrow \langle y\rangle(Tr((Woman(y) \wedge Loves(x,y))))) \\
[x](x{:}Man \rightarrow \langle y\rangle(\ Tr(Woman(y)) \wedge Tr(Loves(x,y)))) \\
[x](x{:}Man \rightarrow \langle y\rangle(y{:}Woman \wedge x{:}\langle Loves\rangle y))
\end{array}
$$

The Other Translation

- Of course, we can do the translation in the other direction as well.
- We only need to realize that the semantic definition of all the operators we introduced can be defined in first-order logic.

$$\mathcal{M}, w \models \langle R\rangle\varphi \quad \text{iff} \quad \text{there is } w' \text{ s.t. } wRw' \text{ and } \mathcal{M}, w' \models \varphi$$

$$Tr_w(\langle R\rangle\varphi) \quad = \quad \exists w'.(R(w,w') \wedge Tr_{w'}(\varphi))$$

- The w in Tr_w keeps track of where we are evaluating the formula in the model.

The Other Translation

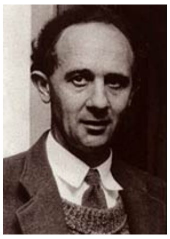
- Let's see the details. Assume that we have a formula in the $\langle R\rangle$ language extended with constants, and the the : and $\langle n\rangle$ operators.
  We will (recursively) define an equivalent first order formula:

$$
\begin{array}{rcl}
Tr_x(p) & = & P(x) \\
Tr_x(n_i) & = & (n_i = x) \\
Tr_x(\neg\varphi) & = & \neg Tr_x(\varphi) \\
Tr_x(\varphi \wedge \psi) & = & Tr_x(\varphi) \wedge Tr_x(\psi) \\
Tr_x(\langle R\rangle\varphi) & = & \exists x.(R(x,y) \wedge Tr_y(\varphi)) \quad \text{for } y \text{ a new variable} \\
Tr_x(n{:}\varphi) & = & Tr_n(\varphi) \\
Tr_x(\langle n\rangle\varphi) & = & \exists n. Tr_x(\varphi)
\end{array}
$$

What we Covered Today

- In a way, the reason for today's talk was to show that there is nothing special about first order logic.
- It can be obtained in a natural way, following the ideas that we introduced in previous lectures.
- People has told me
  I undestand PL, but I would never get how FOL works.
  NONSENSE!!
- As we saw today, they are not as different. You only need to look at them from the right perspective.
- If you really understand how one works, you already know how the other does.

Relevant Bibliography



- As Patrick mentioned in the first lecture one of the first polytheistic logicians was Arthur Prior.
- Prior is the father of Tense Logic, a logic that include the operators F and P to talk about the future and the past.
- He was a strong advocate of the bottom up way of viewing first-order logic that we presented today.

Prior, Arthur (1967). *Chapter V.6 of Past, Present and Future*. Clarendon Press, Oxford.

The Next Lecture

We Like it Complete and Compact

(and We have a Soft Spot for Löwenheim-Skolem)