Logics for Computation

Lecture #7: DIY First-Order Logic

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INRIA Nancy Grand Est
Nancy, France

ESSLLI 2008 - Hamburg - Germany

The Story so Far

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- What do you think? Can we mix the First Order Recipe?

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- We'll cook First Order Logic à la ESSLLI08.
- We will see what we can reuse of what we already have...
- ...and extend the language if necessary.
- We will then show that the language we obtain is actually equivalent to the 'classical' First Order Language.

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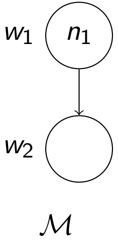

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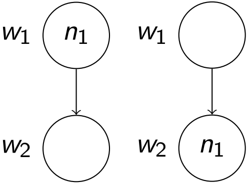

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- Actually, using $\langle n \rangle$ and : together we can define $[U]$:

$$[U]\varphi \text{ iff } \neg \langle n \rangle (n{:}\neg \varphi)$$

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- If you really understand how one works, you already know how the other does.

Relevant Bibliography

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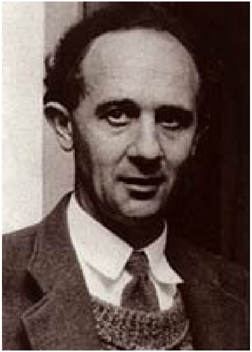

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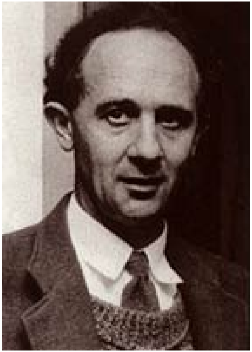

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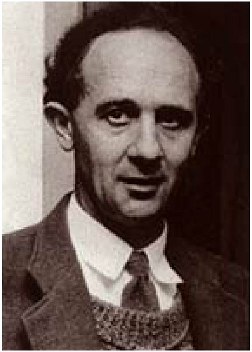

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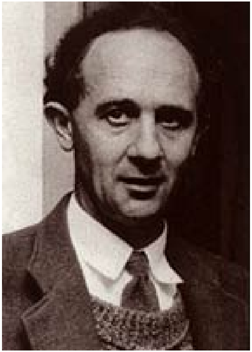

Prior, Arthur (1967). *Chapter V.6 of Past, Present and Future*. Clarendon Press, Oxford.

The Next Lecture

We Like it Complete and Compact

(and We have a Soft Spot for Löwenheim-Skolem)