Logics for Computation

Lecture #9: Putting Tiles in an Infinite Bathroom

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ESSLLI 2008 - Hamburg - Germany

The Story so Far

- We are working with first-order logic.
- We discussed soundness and completeness
  Intuitively, they are the way to synchronize the semantics of a logic with an inferece method like tableaux.
  They are the way to show that we got it right.
- We also discussed some properties like Compactness and Löwenhein-Skolem.
- We saw that this properties actually characterize first order logic: Lindström Theorem.
- We've come a long way. Think that on Monday we were in propositional logic!

What do we do Today

- We'll talk about the satisfiability problem of first-order logic.
- We will argue that the problem is undecidable
  - That is, there is no algorithm that can answer for any formula of first order logic, whether the formula has a model or not.
- Actually, we are going to show how to tile and infinite bathroom.

Tableaux for First Order Logic

- Patrick introduced these two rules yesterday:

$$\frac{s{:}\langle x\rangle\varphi}{s{:}\varphi[x \leftarrow n]}\ (\langle x\rangle) \qquad \frac{s{:}\neg\langle x\rangle\varphi}{s{:}\neg\varphi[x \leftarrow o]}\ (\neg\langle x\rangle)$$

(where n is a new name) (where o is an old name)

- How comes that this tableaux do not terminate?
- As we already mentioned,
  The SAT problem of FO is undecidable.
- Hence, we cannot expect any sound and complete tableaux to also terminate.

(Un)Decidability

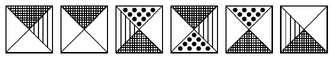
How can we prove that problem X is undecidable?
One way is

- Ask somebody (more intelligent than us) to prove that some problem Y is undecidable
- Prove that if X would be decidable then Y would be decidable, giving a codification of Y into X.

The halting problem of Turing machines is the standard example of an undecidable problem. The behaviour of a Turing machine, and the predicate that says that a given turing machine stops on all inputs can be expressed in FO.

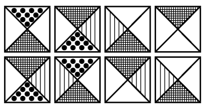
Tiling Problems

- A tiling problem is a kind of jigsaw puzzle
- a tile T is a 1×1 square, fixed in orientation, with a fixed color in each side
- for example, here we have six different kinds of tiles:



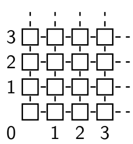
- a simple tiling problem, could be:

Is it possible to place tiles of the kind we show above on a grid of 2×4, in such a way that we cover the entire grid and that adjacent tiles have the same color on neighboring sides?



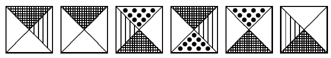
Tiling Problems

- The general form of a tiling problem
  Given a finite number of kind of tiles $\mathcal{T}$, can we cover a given part of $\mathbb{Z} \times \mathbb{Z}$ in such a way that adjacent tiles have the same color on the neighboring sides?
- In some cases, it is also possible to impose certain conditions on what is considered a correct tiling.
- Covering $\mathbb{N} \times \mathbb{N}$
  - tiling $\mathbb{N} \times \mathbb{N}$: Given a finite set of tiles $\mathcal{T}$, can $\mathcal{T}$ cover $\mathbb{N} \times \mathbb{N}$?
  - this problem is undecidable (It is equivalent to the halting problem of Turing machines)



Representing Tiling Problems

- Notice that every finite set of tiles $\mathcal{T} = \{t_1, \ldots, t_k\}$ can be represented as two binary relations H, and V.
  We put $H(t_i, t_j)$ when the right side of t_i coincide with the left side of t_j, and similarly for V.
- For example, for



t_1 t_2 t_3 t_4 t_5 t_6

we will have

$$\begin{aligned} H &= \{(t_1,t_3),(t_1,t_6),(t_2,t_4),(t_2,t_5),(t_2,t_1),\ldots\} \\ V &= \{(t_1,t_3),(t_1,t_5),(t_1,t_6),(t_2,t_3),(t_2,t_5),\ldots\} \end{aligned}$$

Coding a Tiling of the Grid in FO

- Theorem: The problem of deciding whether a given FO formula is satisfiable is undecidable.
  Let $\rightarrow$ and $\uparrow$ be relations, and $t_1, \ldots, t_n$ be propositinal symbols.
  Let φ be the conjunction of the formulas:

  1) Total: $[x]\langle y\rangle(x{:}\langle R\rangle y)$ for $R \in \{\rightarrow, \uparrow\}[x]\langle y\rangle(x{:}\langle R\rangle y)$ for $R \in \{\rightarrow, \uparrow\}$
  2) Functional: $[x][y][z](x{:}\langle R\rangle y \wedge x{:}\langle R\rangle z \rightarrow y{:}z)$ for $R \in \{\rightarrow, \uparrow\}[x][y][z](x{:}\langle R\rangle y \wedge x{:}\langle R\rangle z \rightarrow y{:}z)$ for $R \in \{\rightarrow, \uparrow\}$
  3) Commuting:
     $[x][y](x{:}\langle\rightarrow\rangle\langle\uparrow\rangle y \leftrightarrow x{:}\langle\uparrow\rangle\langle\rightarrow\rangle y)[x][y](x{:}\langle\rightarrow\rangle\langle\uparrow\rangle y \leftrightarrow x{:}\langle\uparrow\rangle\langle\rightarrow\rangle y)$
  4) Tiled: $[x](x{:}t_1 \vee \ldots \vee x{:}t_n)[x](x{:}t_1 \vee \ldots \vee x{:}t_n)$,
  5) But not Twice:
     $[x](x{:}t_i \rightarrow x{:}\neg t_j)$ for $i \neq j$ $[x](x{:}t_i \rightarrow x{:}\neg t_j)$ for $i \neq j$,
  6) Horizontal Match: $[x][y]((x{:}t_i \wedge x{:}\langle\rightarrow\rangle y) \rightarrow (\bigvee_{H(t_i,t_j)} y{:}t_j))[x][y]((x{:}t_i \wedge x{:}\langle\rightarrow\rangle y) \rightarrow (\bigvee_{H(t_i,t_j)} y{:}t_j))$,

Relevant Bibliography I

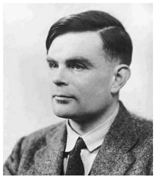
Undecidable Problems

- Alonzo Church invented lambda calculus and propose it as a model for computation.
- He showed then that the problem of satisfiability of first-order logic could not be coded in the lambda calculus. Hence it was uncomputable.
- Here is a list of some of Church's doctoral students: A. Anderson, P. Andrews, M. Davis, L. Henkin, S. Kleene, M. Rabin, B. Rosser, D. Scott, R. Smullyan, and A. Turing.

Church, Alonzo (1956). *Introduction to Mathematical Logic*. The Princeton University Press.

Relevant Bibliography II

Undecidable Problems



- Alan Turing invented Turing Machines and Computer Science.
- He showed that the halting problem could not be decided by a Turing Machine.
- And that the behavior of a Turing Machine can easily be described in first-order logic, providing an alternative proof that the satisfiability problem of first-order logic is undecidable.

Turing, Alan (1936), *On Computable Numbers, with an Application to the Entscheidungsproblem*. Proceedings of the London Mathematical Society, Series 2, Vol.42, pp 230–265, `http://www.thocp.net/biographies/papers/turing_oncomputablenumbers_1936.pdf`