Logics for Computation

Lecture #9: Putting Tiles in an Infinite Bathroom

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- We also discussed some properties like Compactness and Löwenhein-Skolem.
- We saw that this properties actually characterize first order logic: Lindström Theorem.
- We've come a long way. Think that on Monday we were in propositional logic!

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- We'll talk about the satisfiability problem of first-order logic.
- We will argue that the problem is undecidable
  - That is, there is no algorithm that can answer for any formula of first order logic, whether the formula has a model or not.
- Actually, we are going to show how to tile and infinite bathroom.

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- Patrick introduced these two rules yesterday:

$$\frac{s{:}\langle x\rangle\varphi}{s{:}\varphi[x \leftarrow n]}\ (\langle x\rangle) \qquad \frac{s{:}\neg\langle x\rangle\varphi}{s{:}\neg\varphi[x \leftarrow o]}\ (\neg\langle x\rangle)$$

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- How comes that this tableaux do not terminate?
- As we already mentioned,

  The SAT problem of FO is undecidable.

- Hence, we cannot expect any sound and complete tableaux to also terminate.

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The halting problem of Turing machines is the standard example of an undecidable problem. The behaviour of a Turing machine, and the predicate that says that a given turing machine stops on all inputs can be expressed in FO.

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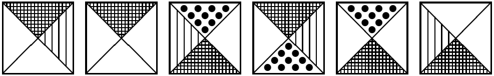

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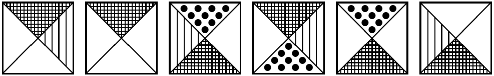

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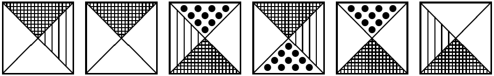

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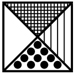

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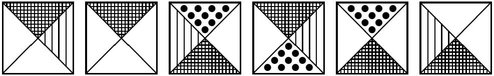

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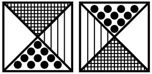

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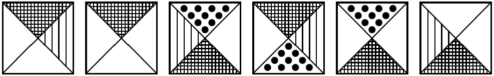

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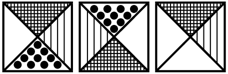

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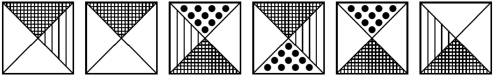

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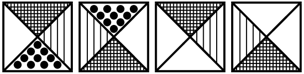

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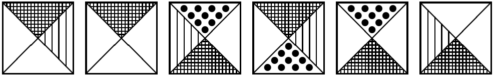

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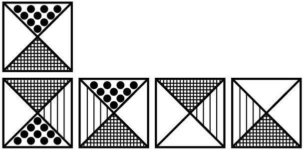

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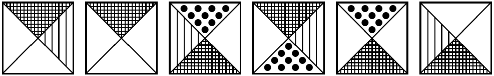

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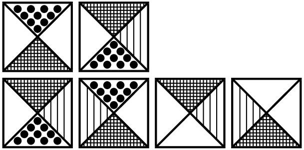

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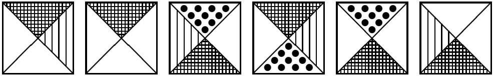

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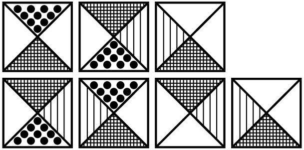

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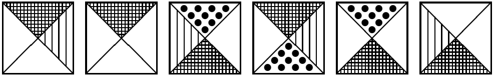

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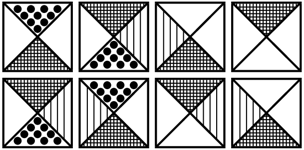

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  Given a finite number of kind of tiles $\mathcal{T}$, can we cover a given part of $\mathbb{Z} \times \mathbb{Z}$ in such a way that adjacent tiles have the same color on the neighboring sides?

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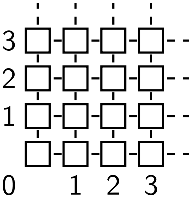
- Covering $\mathbb{N} \times \mathbb{N}$
  - tiling $\mathbb{N} \times \mathbb{N}$: Given a finite set of tiles $\mathcal{T}$, can $\mathcal{T}$ cover $\mathbb{N} \times \mathbb{N}$?
  - this problem is undecidable (It is equivalent to the halting problem of Turing machines)



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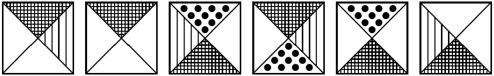

t_1 t_2 t_3 t_4 t_5 t_6

we will have

$$\begin{aligned} H &= \{(t_1, t_3), (t_1, t_6), (t_2, t_4), (t_2, t_5), (t_2, t_1), \ldots\} \\ V &= \{(t_1, t_3), (t_1, t_5), (t_1, t_6), (t_2, t_3), (t_2, t_5), \ldots\} \end{aligned}$$

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  Then φ is satisfiable iff $\mathcal{T}$ covers $\mathbb{N} \times \mathbb{N}$.

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Church, Alonzo (1956). *Introduction to Mathematical Logic*. The Princeton University Press.

Relevant Bibliography II

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- Alan Turing invented Turing Machines and Computer Science.

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Turing, Alan (1936), *On Computable Numbers, with an Application to the Entscheidungsproblem*. Proceedings of the London Mathematical Society, Series 2, Vol.42, pp 230–265, http://www.thocp.net/biographies/papers/turing_oncomputablenumbers_1936.pdf