Logics and Statistics for Language Modeling

Carlos Areces

areces@loria.fr
http://www.loria.fr/~areces/ls
INRIA Nancy Grand Est
Nancy, France

2009/2010

Today's Program

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- Propositional Logics (Again)
  - Syntax and Semantics
  - Satisfiability, Validity, Contingency, etc.
  - The Tableau Method
  - Decision Methods
  - Exercises

Propositional Logic

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- The language of propositional logic (PL) is very easy:

Some propositional symbols:	$p_1, p_2, p_3, \ldots$
Two logical symbols:	$\neg, \vee$
Two syntactic symbols:	(,)

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- What is a formula?
  - Every propositional symbol p_i is a formula in PL
  - If φ is a formula in PL then $\neg(\varphi)$ is a formula in PL
  - If φ and ψ are formulas in PL then $(\varphi \vee \psi)$ is a formula in PL
  - Nothing esle is a formula

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$$\begin{array}{rcl} V \models \neg(\phi) & \text{iff} & V \not\models \phi \\ V \models (\phi \vee \psi) & \text{iff} & \text{either } V \models \phi \text{ or } V \models \psi \end{array}$$

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- We say that a PL formula φ is satisfiable iff for some V, $V \models \varphi$.
- We say that a PL formula φ is unsatisfiable iff it is not satisfiable.
- We say that a PL formula φ is a tautology iff for all V, $V \models \varphi$.
- We say that a PL formula φ is a contradiction iff for no V, $V \models \varphi$.
- We say that a PL formula φ is a contingency iff it is neither a tautology nor a contradiction.

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So let's turn to satisfiability checking ...

- We'll use tableaux to perform this task.
- A tableaux is essentially a tree-like data structure that records attempts to build a model.
- Tableaux are built by applying rules to an input formula. These rules systematically tear the formula to detect all possible ways of building a model.
- Each branch of a tableaux records one way of trying to build a model. Some branches ("closed branches") don't lead to models. Others branching ("open branches") do.
- The best way to learn is via an example...

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p \\
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r
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Contradiction!!!	*Model*

Formalizing the Diplomatic Problem

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- Three propositional symbols

$$\begin{array}{lcllcl} P & \equiv & \text{invite Peru} & \neg P & \equiv & \text{exclude Peru} \\ Q & \equiv & \text{invite Qatar} & \neg Q & \equiv & \text{exclude Qatar} \\ R & \equiv & \text{invite Romania} & \neg R & \equiv & \text{exclude Romania} \end{array}$$

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- The problem can be formalized as

prince: $P \vee \neg Q$	$\equiv$	invite Peru or exclude Qatar
queen: $Q \vee R$	$\equiv$	invite Qatar or Romania or both
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- $\Sigma = (P \vee \neg Q) \; \& \; (Q \vee R) \; \& \; (\neg R \vee \neg P)$

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Contradiction!!!

It is impossible to apply any more rules, and there are no open branches. Hence no model exists for the input $\neg\varphi$. Hence φ is valid.

Some of the Techniques for SAT Solving

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- Complete methods (they answer SAT if and only if the formula is satisfiable).
  - truth tables
  - tableaux
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  - resolution
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- Approximation procedures (they answer SAT or UNKOWN)
  - flip the value of a variable in an unsatisfied clause
  - genetic algorithms
  - hill-climbing

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  - They always answer SATISFIABLE or UNSATISFIABLE.
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- The best known decision methods probably are
  - truth tables
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Graph Coloring: The Nitty-Gritty Details

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   for $1 \leq i \leq n$, and $1 \leq l < m \leq k$
3. Neighboring nodes have different colors. $\neg p_{il} \vee \neg p_{jl}$,
   for i and j neighboring nodes, and $1 \leq l \leq k$

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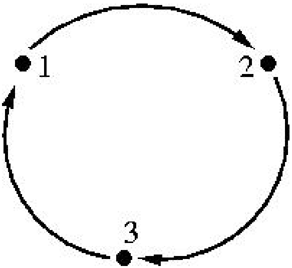
- An application to algebra:
  - problems dealing with quasigroups can be viewed as specialized coloring problems
  - some open problems on quasigroups have been encoded in this way and solved automatically by propositional provers. E.g. is there a quasigroup satisfying the equations

$$\forall a.(a \cdot a) = a$$
$$\forall a, b.((b \cdot a) \cdot b) \cdot b = a$$

Exercises

Use the graph coloring coding in the following graph, using 2 colors (R and B):



Show the first three lines of a truth table for the formula obtained in the codification.