Logics and Statistics for Language Modeling

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2009/2010

Today's Program

- First Order Logics.
  - Syntax
  - Models
  - Semantics

First Order Logic

While propositional logic assumes that the world contains facts, first-order logic (like natural language) assumes the world contains

- Objects: or elements in the world like people, houses, numbers, cars, windows, doors, cups, trees, cats, ...
- Functions: (one to one mappings) defined over objects in the world like father of, best briend, one more than, plus, ...
- Relations: between or about objects in the world like red, round, prime, brother of, bigger than, part of, taller than,...

Syntax of FOL: The Logical Language

In FOL we use a logical language which has a fixed interpretation:

- Boolean Connectives: $\neg$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$
- Quantifiers: $\forall$, $\exists$
- Equality: $=$
- Punctuation:), (, .

Syntax of FOL: Basic elements

A first-order language is defined in terms a set of constant symbols variables, predicate symbols, function symbols (a signature):

- Constant symbols: π, 2, Carlos, ID223, B230, LORIA, etc.
  They will stand by special elements in a given situation.
- Variables: x, y, z, a, b, etc.
  They stand for arbitrary objects in a given situation.
- Function symbols: Sqrt, LeftLegOf, LengthOf, Succ, etc.
  They will be the names of the functions in a given situation.
- Predicate symbols: Brother, $\geq$, Tall, etc.
  They will be the particular relations among or about the elements in a given situation.

These are also called the Non Logical Language and we should specify their meaning in each particular situation.

Syntax of FOL: Signature

A signature is a 4-uple $S = \langle VAR, CONS, RELS, FUNS\rangle$, where

VAR is a set of variables (e.g., $VAR = \{x, y, z\}$)
$CONS$ is a set of constant symbols (e.g., $CONS = \{3, carlos, a\}$)
$RELS$ is a set of relational symbols
(e.g. $RELS = \{R^3, BrotherOf^2, Tall^1\}$)
$FUNS$ is a set of function symbols (e.g. $FUNS = \{f^1, +^2\}$).

Elements in the signature will be our basic vocabulary when writing down FO formulas.

Syntax of FOL: Terms

Assume we are given a signature $S = \langle VAR, CONS, RELS, FUNS\rangle$.

Terms:

a constant symbol	E.g.: Carlos, Graciela
a variable	E.g.: x
function symbol(term$_1$,..., term$_n$)	E.g.: LeftLegOf(Carlos)

Terms can be seen as 'complex names' for elements in the situation.

Examples and Non-Examples: Consider the signature in the previous slide:

$VAR = \{x, y, z\}$
$CONS = \{3, carlos, a\}$
$RELS = \{R^3, BrotherOf^2, Tall^1\}$
$FUNS = \{f^1, +^2\}$

x	YES
f	NO
$f(x)$	YES
$+(3)$	NO
$Tall(carlos)$	NO

Syntax of FOL: Atomic Formulas

Atomic formulas are the basic units about which we can claim truth or falsity in a given situation.

Atomic Formulas:
(term$_1$ = term$_2$)
E.g. Length(LeftLegOf(Carlos)) = Lenght(RightLegOf(Carlos))
predicate symbol(term$_1$, ..., term$_2$)
E.g. Brother(Carlos,Graciela)

Examples: The following are atomic formulas (over the appropriate signature)

$(+(2, 2) = 4)$
$<(x, 2)$
$f(y)$
$P(x, \pi, carlos, 4, xyz)$

Which is the appropriate signature?:
$VAR = \{x, y\}$
$CONS = \{2, 4, \pi, carlos, xyz\}$
$FUNS = \{+^2\}$
$RELS = \{<^2, f^1, P^5\}$

Syntax of FOL: Complex Formulas

Complex formulas are built up from atomic formulas using the boolean connectives and the quantifiers:

$$FORMS := ATOM \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \exists x.(\varphi)$$

where φ is a formula, and x is a variable.
($\forall$ is defined in terms of $\neg$ and $\exists$: $\forall x.(\varphi) \equiv \neg\exists x.(\neg\varphi)$.)

Examples:

((Brother(carlos,graciela) ∧ Woman(graciela))→ Sister(graciela,carlos))

$$\forall x.(\forall y.(>(x,y) \vee ((x=y) \vee >(y,x))))$$

FOL: Semantics (Warning: Most Complex Slide Today!)

- FOL formulas are true or false with respect to a model and an assignment for the model.
- A model is a pair $\langle D, I\rangle$ where
  - D is the domain: a non empty set of objects
  - I is the interpretation: a function assigning *meaning* to constant symbols, relation symbols and function symbols.
- And assignment is a function that to each variable assigns an element of D.

Formally, given a signature $S = \langle VAR, CONS, RELS, FUNS\rangle$ a model for S is $M = \langle D, I\rangle$ such that,

- for $c \in CONS$, $I(c) \in D$
- for $R \in REL$ n-ary, $I(R) \subseteq D^n$ (an n-ary relation in D)
- for $F \in FUN$ n-ary, $I(F) : D^n \mapsto D$ (an n-ary function in D)

An assignment g for M is a function $g : VAR \mapsto D$.

Satisfiability

- Given a model $M = \langle D, I\rangle$ and an assignment g for M we want to define when a given FOL formula φ is true or false in M, g ($M, g \models \varphi$). We do this case by case.
- We first define the meaning of complex terms. The interpretation gives us the meaning for constants ($I(c)$) and funcions ($I(F)$). The assignment gives us the meaning of variables ($g(x)$) We can then define:

$$\begin{aligned} x^{I,g} &= g(x) \\ c^{I,g} &= I(c) \\ (f(t_1,\ldots,t_n))^{I,g} &= I(f)(t_1^{I,g},\ldots,t_n^{I,g}) \end{aligned}$$

- Now we can define $M, g \models \varphi$ for arbitrary formulas...

Satisfiability

- $M, g \models t_1 = t_2$ if and only if $t_1^{I,g} = t_2^{I,g}$
- $M, g \models R(t_1, \ldots t_n)$ if and only if $(t_1^{I,g}, \ldots, t_n^{I,g}) \in I(R)$
- $M, g \models \neg\varphi$ if and only if $M, g \not\models \varphi$
- $M, g \models (\varphi_1 \wedge \varphi_2)$ if and only if $M, g \models \varphi_1$ and $M, g \models \varphi_2$
- $M, g \models \exists x.(\varphi)$ if and only if $M, g' \models \varphi$ for some assignment g' identical to g excepts perhaps in $g(x)$.

Exercises

- Define an adequate signature and write first order formulas for the following sentences.
  1. There is one triangle and two circles.
  2. Each object has a color: either red, blue or green.
  3. Circles are neither squares nor green.
  4. Circles are bigger than triangles.
- Give a formal definition of a model $\mathcal{M}$ s.t.:
  - $\mathcal{M}$ is a proper model for the signature used in the formulas above.
  - All the formulas above are true in $\mathcal{M}$.