Logics and Statistics for Language Modeling

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Today's Program

- Resolution for FOL
  - Clausal Form. Skolemization.
  - The Resolution Rules
  - Non Termination

Some Properties of Quantifiers

- $\forall x.\forall y.\varphi$ is the same as $\forall y.\forall x.\varphi$
- $\exists x.\exists y.\varphi$ is the same as $\exists y.\exists x.\varphi$
- $\exists x.\forall y.\varphi$ is not the same as $\forall y.\exists x.\varphi$
- $\forall x.\varphi$ is the same as $\forall y.\varphi[x/y]$ if y does not appear in φ, and similarly for $\exists x.\varphi$ and $\exists y.\varphi[x/y]$.
- $\varphi \wedge Qx.\psi$ is the same as $Qx.(\varphi \wedge \psi)$ if x does not appear in φ ($Q \in \{\forall, \exists\}$).
- $\neg\exists x.\varphi$ is equivalent to $\forall x.\neg\varphi$ and $\neg\forall y.\varphi$ is equivalent to $\exists x.\neg\varphi$.

Prenex Normal Form

- **Prenex Normal Form**
  - Write φ in negation normal form.
  - Use different variables for all bounded variables (each variable should appear either bound or free, and each quantifier should use a different variable).
  - Move quantifiers to the front without changing alternation.
- **Example**

$$\forall x.\neg(\exists y.R(x,y) \wedge \forall y.R(x,y))$$
$$\forall x.(\neg\exists y.R(x,y) \vee \neg\forall y.R(x,y))$$
$$\forall x.(\forall y.\neg R(x,y) \vee \exists y.\neg R(x,y))$$
$$\forall x.(\forall y.\neg R(x,y) \vee \exists z.\neg R(x,z))$$
$$\forall x.\forall y.\exists z.(\neg R(x,y) \vee \neg R(x,z))$$

Clausal Form and Skolemnization

- Write φ in prenex normal form (PNF), with the matrix in conjunctive normal form:
  $\varphi = \mathcal{Q}.\psi$ where $\psi = \bigwedge_{l \in L} \bigvee_{m \in M} \psi_{(l,m)}$
- Let $Sko(\varphi)$ be the "skolemnization" of φ, obtained as follows
  - While there is an existential quantifier in $\mathcal{Q}$, let $\bar{x}$ be the list of variables universally quantified in $\mathcal{Q}$ which occur in front of the first existential quantifier $\exists x_i$.
  - Eliminate $\exists x_i$ from $\mathcal{Q}$ and replace ψ by $\psi[f(\bar{x})/x_i]$ where f is a fresh $|\bar{x}|$-ary function not used before.
- After eliminating all the existential quantifiers, drop $\mathcal{Q}$, consider the obtained matrix as a propositional formula in conjunctive normal form and define $ClSet$ as we did before.

Skolemnization Examples

1 $\exists x.\forall y.\exists z.(P(x,y) \wedge P(y,z) \rightarrow P(x,z))$
Sk: $P(c,y) \wedge P(y,f(y)) \rightarrow P(c,f(y))$

2 $\forall x.\forall y.(P(x,y) \rightarrow \exists z.(P(x,z) \rightarrow P(z,y)))$
Sk: $P(x,y) \rightarrow (P(x,f(x,y)) \rightarrow P(f(x,y),y))$

3 $\forall x.\exists y.P(y,y)$
Sk: $P(f(x),f(x))$

4 $\exists x.\forall x.P(x,x)$
Sk: $P(x,x)$

Resolution for First Order Logic

- Let $ClSet^*(\varphi)$ be the smallest set containing $ClSet(\varphi)$ and clause under the (RES) and (FAC) rules:

$$[RES]\ \frac{Cl_1 \cup \{N\} \in ClSet^*(\varphi) \qquad Cl_2 \cup \{\neg M\} \in ClSet^*(\varphi)}{(Cl_1 \cup Cl_2)\theta \in ClSet^*(\varphi)}$$

$$[FAC]\ \frac{Cl \cup \{N, M\} \in ClSet^*(\varphi)}{(Cl \cup \{N\})\theta \in ClSet^*(\varphi)}$$

where θ is the most general unifier of M and N.

- **Important:** Before applying the [RES] rule, rename variables in the clauses so that they don't share any variable.
- **Theorem:** $\forall\varphi$, $ClSet^*\ \varphi$ is inconsistent iff $\{\} \in ClSet^*(\varphi)$.

Example

1. $\neg((\forall x(P(x) \rightarrow Q(x)) \wedge \forall x(\neg Q(x))) \rightarrow \forall x(\neg P(x)))$ (eliminate $\rightarrow$)
2. $\neg(\neg(\forall x(\neg P(x) \vee Q(x)) \wedge \forall x(\neg Q(x))) \vee \forall x(\neg P(x)))$ (push $\neg$ in)
3. $((\forall x(\neg P(x) \vee Q(x)) \wedge \forall x(\neg Q(x))) \wedge \exists x(P(x)))$ (rename)
4. $((\forall x(\neg P(x) \vee Q(x)) \wedge \forall y(\neg Q(y))) \wedge \exists z(P(z)))$ (move to PNF)
5. $\exists z\forall y\forall x(((\neg P(x) \vee Q(x)) \wedge (\neg Q(y))) \wedge (P(z)))$ (skolemize)
6. $(((\neg P(x) \vee Q(x)) \wedge \neg Q(y)) \wedge P(c))$ (write as clauses)
7. $\{\{\neg P(x), Q(x)\}, \{\neg Q(y)\}, \{P(c)\}\}$ (resolve)
8. $\{\{\neg P(x), Q(x)\}, \{\neg Q(y)\}, \{P(c)\}, \{\neg P(z)\}, \{Q(c)\}, \{\}\}$ (UNSAT)

Termination?

Let's consider the formula

$$\exists x \forall y (R(x,y) \rightarrow (P(y) \rightarrow \exists z.(R(y,z) \wedge P(z))))$$

1. $\{\neg R(c,y),\ \neg P(y),\ R(y,f(y))\}$
2. $\{\neg R(c,w),\ \neg P(w),\ P(f(w))\}$

With one resolution step we obtained

3. $\{\neg R(c,c),\ \neg P(c),\ \neg P(f(c)),\ P(f^2(c))\}$
4. $\{\neg R(c,f(w)),\ R(f(w),f^2(w)),\ \neg R(c,w),\ \neg P(w)\}$.

Clauses 2 y 4 resolve to give

5. $\{\neg R(c,f^2(w)),\ R(f^2(w),f^3(w)),\ \neg R(c,f(w)),\ \neg R(c,w),$ $\neg P(w)\}$.

State Explosion

- As the example we just saw shows, generating new clauses is easy.
- Indeed, if we are not careful we can easily generate millions of clauses leading nowhere.
- Notice that every time we generate a formula implied by another formula already in the clause set we are wasting time.
- Discovering when this is happening to be able to avoid it, is where most FO provers spend their computing time (simplification and subsumption)
- The "no redundancy" constraint helps us keep the clause set under control, as we will reach sooner the point of saturation, where no new, non redundant clauses can can be generated.

Exercises

- Apply the resolution method to the following formula, to determine whether it's satisfiable:

$$\forall x.\exists y.(R(x,y) \rightarrow Q(y)) \wedge \forall y.\neg Q(y)$$

- Now try with

$$\forall x.\exists y.(R(x,y) \rightarrow Q(y)) \wedge \forall y.\neg Q(y) \wedge \exists x.R(x,x)$$