Logics and Statistics for Language Modeling

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Today's Program

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- Resolution for FOL
  - Clausal Form. Skolemization.
  - The Resolution Rules
  - Non Termination

Some Properties of Quantifiers

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- $\forall x.\forall y.\varphi$ is the same as $\forall y.\forall x.\varphi$
- $\exists x.\exists y.\varphi$ is the same as $\exists y.\exists x.\varphi$
- $\exists x.\forall y.\varphi$ is not the same as $\forall y.\exists x.\varphi$
- $\forall x.\varphi$ is the same as $\forall y.\varphi[x/y]$ if y does not appear in φ, and similarly for $\exists x.\varphi$ and $\exists y.\varphi[x/y]$.
- $\varphi \wedge Qx.\psi$ is the same as $Qx.(\varphi \wedge \psi)$ if x does not appear in φ ($Q \in \{\forall, \exists\}$).
- $\neg\exists x.\varphi$ is equivalent to $\forall x.\neg\varphi$ and $\neg\forall y.\varphi$ is equivalent to $\exists x.\neg\varphi$.

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- After eliminating all the existential quantifiers, drop $\mathcal{Q}$, consider the obtained matrix as a propositional formula in conjunctive normal form and define $ClSet$ as we did before.

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- Let $ClSet^*(\varphi)$ be the smallest set containing $ClSet(\varphi)$ and clause under the (RES) and (FAC) rules:

$$[RES]\ \frac{Cl_1 \cup \{N\} \in ClSet^*(\varphi) \qquad Cl_2 \cup \{\neg M\} \in ClSet^*(\varphi)}{(Cl_1 \cup Cl_2)\theta \in ClSet^*(\varphi)}$$

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  where θ is the most general unifier of M and N.
- **Important:** Before applying the [RES] rule, rename variables in the clauses so that they don't share any variable.
- **Theorem:** $\backslash\forall\varphi$, $ClSet^*\ \varphi$ is inconsistent iff $\{\} \in ClSet^*(\varphi)$.

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Clauses 2 y 4 resolve to give

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- Discovering when this is happening to be able to avoid it, is where most FO provers spend their computing time (simplification and subsumption)
- The "no redundancy" constraint helps us keep the clause set under control, as we will reach sooner the point of saturation, where no new, non redundant clauses can can be generated.

Exercises

- Apply the resolution method to the following formula, to determine whether it's satisfiable:

$$\forall x. \exists y. (R(x, y) \rightarrow Q(y)) \land \forall y. \neg Q(y)$$

- Now try with

$$\forall x. \exists y. (R(x, y) \rightarrow Q(y)) \land \forall y. \neg Q(y) \land \exists x. R(x, x)$$