Worksheet 1

Modal Logics: A Modern Perspective

Stanford University, Spring Term, 2018

1. Review

Exercise 1.1. Given a formula φ in the basic modal language, we define the set of positive, negative and all propositional symbols ($Pos(\varphi)$, $Neg(\varphi)$ y $Prop(\varphi)$ respectively) as follows:

φ	$Prop(\varphi)$	$Pos(\varphi)$	$Neg(\varphi)$
p	$\{p\}$	$\{p\}$	$\emptyset$
$\neg\varphi$	$Prop(\varphi)$	$Neg(\varphi)$	$Pos(\varphi)$
$\varphi \wedge \psi$	$Prop(\varphi) \cup Prop(\psi)$	$Pos(\varphi) \cup Pos(\psi)$	$Neg(\varphi) \cup Neg(\psi)$
$\langle R\rangle\varphi$	$Prop(\varphi)$	$Pos(\varphi)$	$Neg(\varphi)$

(a) Prove (by induction) that for any formula φ

$$Prop(\varphi) = Pos(\varphi) \cup Neg(\varphi)$$

and give an example where $Pos(\varphi) \cap Neg(\varphi) \neq 0$.

Exercise 1.2. Given a formula φ, we say that $Sub(\varphi)$ is the set of all subformulas of φ.

(a) Give a recursive definition of Sub for the basic modal language and for PDL with tests.

(b) Write down the set $Sub([R]([R]p \to p) \to [R]p)$.

(c) Prove that if $\psi \in Sub(\varphi)$ then $Sub(\psi) \subseteq Sub(\varphi)$ for any ψ and φ.

2. Modal Language

Exercise 2.1. If $K\varphi$ means "the agent knows that φ" and $M\varphi$ means "it is consistent with what the agent knows that φ", write down formulas that represent the following statements:

(a) If φ is true, then it is consistent with the knowledge of the agent that (s)he knows φ.

(b) If it is consistent with the knowledge of the agent that φ, and it is consistent with his/her knowledge that ψ, then it is consistent with his/her knowledge that $\varphi \wedge \psi$.

(c) If the agent knows φ, then it is consistent with what (s)he knows that φ.

(d) If it is consistent with what the agent knows that it is consistent with what (s)he knows that φ, then it is consistente with what (s)he knows that φ.

Which of this statements "make sense"?

Exercise 2.2. Suppose that $\Diamond\varphi$ is interpreted as "φ is permited. How should $\Box\varphi$ be interpreted? List some formulas which seems "plausible" under this interpretation.

Exercise 2.3. Consider PDL with tests. Write down formulas that could represent:

(a) `while` φ `do` ψ
(b) `repeat` φ `until` ψ

Exercise 2.4. The basic temporal language has two modalities, $\langle F\rangle$ and $\langle P\rangle$. The interpretation of $\langle F\rangle\varphi$ is "φ is going to be true at some future time", while $\langle P\rangle\varphi$ means "φ was true at some past time". Decide if the following formulas should be valid under this interpretation[1]:

(a) $\langle F\rangle\varphi \to \langle F\rangle\langle F\rangle\varphi$
(b) $\langle F\rangle\varphi \to \langle P\rangle\langle F\rangle\varphi$
(c) $\neg\langle F\rangle\neg\varphi \to \langle P\rangle\langle F\rangle\varphi$
(d) $\langle P\rangle\varphi \to \langle F\rangle\neg\langle P\rangle\neg\varphi$

3. Satisfiability - Kripke Models

Exercise 3.1. Show that when evaluationg a formula φ in a model, only the propositional symbols appearing in φ are relevant. I.e., show that given two models $\mathcal{M} = \langle W, \{R_i\}, V\rangle$ and $\mathcal{M}' = \{W, \{R_i\}, V'\}$ s.t. $V(p) = V'(p)$ for all propositional symbols in φ, then for any w, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w \models \varphi$.

Exercise 3.2. Let $\mathcal{N} = \langle \mathbb{N}, \{S_1, S_2\}, V_{\mathcal{N}}\rangle$ and $\mathcal{B} = \langle \mathbb{B}, \{R_1, R_2\}, V_{\mathcal{B}}\rangle$ be two models for a modal language with two modalities $\Diamond_1$ and $\Diamond_2$. $\mathbb{N}$ is the set of natural numbers, and $\mathbb{B}$ is the set of all strongs containing only 0s and 1s. The accessibility relations are defined as:

$$\begin{array}{rll} mS_1n & \text{iff} & n = m+1 \\ mS_2n & \text{iff} & m > n \\ sR_1t & \text{iff} & t = s0 \text{ or } t = s1 \\ sR_2t & \text{iff} & t \text{ is a proper initial segment of } s \end{array}$$

Which of the following formulas are true over $\mathcal{N}$ and $\mathcal{B}$ with respect to an arbitrary valuation?

(a) $(\Diamond_1 p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge q)$
(b) $(\Diamond_2 p \wedge \Diamond_2 q) \to \Diamond_2(p \wedge q)$
(c) $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \to (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$
(d) $p \to \Diamond_1\Box_2 p$
(e) $p \to \Diamond_2\Box_1 p$
(f) $p \to \Box_1\Diamond_2 p$
(g) $p \to \Box_2\Diamond_1 p$

Exercise 3.3. Consider the basic temporal language (se Exercise 2.4, including footnote) and the models $(\mathbb{Z}, <, V_1)$, $(\mathbb{Q}, <, V_2)$ and $(\mathbb{R}, <, V_3)$. Define $E\varphi$ as an abbreviation of $P\varphi \vee \varphi \vee F\varphi$ and $A\varphi$ to represent $H\varphi \wedge \varphi \wedge G\varphi$. Which of the following formulas are true for an arbitrary valuation in each of the models?

(a) $GGp \to p$
(b) $(p \wedge Hp) \to FHp$
(c) $(Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)) \to E(Hp \wedge G\neg p)$

Exercise 3.4. Prove that the following formulas of the basic modal language are not valid, showing a model where they are false.

(a) $\Box\bot$
(b) $\Diamond p \to \Box p$
(c) $p \to \Box\Diamond p$

[1]Historically, these modalities were written as $F\varphi$ and $P\varphi$ and its duals as $G\varphi$ and $H\varphi$.

(d) $\Diamond\Box p \to \Box\Diamond p$

Exercise 3.5. Demostrar que en todo modelo donde la relación de accesibilidad es transitiva, las siguientes fórmulas son verdaderas para cualquier valuación arbitraria:

(a) $\Box\Diamond\Diamond p \to \Box\Diamond p$
(b) $\Box\Diamond\Box\Diamond p \to \Box\Diamond p$

Exercise 3.6. Prove that the formula

$$\Diamond(i \wedge q) \wedge \Diamond(i \wedge p) \to \Diamond(p \wedge q)$$

in the basic hybrid logic is valid. Show a counter-example when i is replaced by a propositional symbol.

Exercise 3.7. Prove that in the hybrid logic $\mathcal{HL}(@)$ the @ operator defines a congruence relation. I.e, that the following formulas are valid:

- $\models @_i i$.
- If $\models @_i j$ then $\models @_j i$.
- If $\models @_i j$ and $\models @_j k$ then $\models @_i k$.
- If $\models @_i j$ then $\models @_i\varphi \leftrightarrow @_j\varphi$.

Exercise 3.8. Consider the logic $\mathcal{HL}(A)$, which is the basic modal logic extended with nominals and the universal operator A. Give a definition of @ in $\mathcal{HL}(A)$.