# Worksheet 2

Modal Logics: A Modern Perspective

Stanford University, Spring Term, 2018

**Exercise 1.** Prove that the following are equivalent:

(a) If $\Gamma \models \varphi$ then for some finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \models \varphi$.

(b) If every finite subset $\Gamma_0 \subseteq \Gamma$ is satisfiable, then $\Gamma$ is satisfiable.

(c) If $\Gamma$ is unsatisfiable, then some finite subset of $\Gamma$ is unsatisfiable.

**Exercise 2.** Let $ST$ be the standard (not optimized, i.e., without reusing variables) translation that maps formulas of the *basic modal language* into formulas of first-order logic.

(a) Prove that for any model $\mathcal{M}$, any state $w$ in $\mathcal{M}$ and any assignment $g$ it holds

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}, g[x \mapsto w] \models ST_x(\varphi).$$

(b) Show that for any $\varphi$, $ST_x(\varphi)$ has $x$ and only $x$ free.

**Exercise 3.** Extend the standard translation $ST$ for the hybrid logic $\mathcal{HL}(@)$. More formally, first, determine the correspondence between first-order models and hybrid models, define the correspondence language and give a translation for the following formulas (with $i \in \textsc{nom}$):

(a) $i$

(b) $@_i\varphi$

Given the results discussed in class, which properties can be transferred to $\mathcal{HL}(@)$. What can we say about decidability?

**Exercise 4.** Decide which of the following properties of FOL can be transferred to the basic modal logic using the standard translation. If you believe that the property cannot be transferred, just state it. If you believe that the property can be transferred, then prove it.

(a) *Connected graph*: There is no formula $\varphi$ s.t. $\mathcal{M} \models \varphi$ iff $\mathcal{M}$ is a connected graph.

(b) *Infinite model*: There is a formula $\varphi$ s.t. if $\mathcal{M} \models \varphi$ then $\mathcal{M}$ is infinite.

**Exercise 5.** Give a translation that preserve *satisfiability* of the basic modal logic over the class of transitive models and prove it is correct. I.e. for any formula $\varphi$ of the basic modal logic, $\varphi$ is satisfiable in a transitive model iff its translated first-order formula $\varphi'$ is satisfiable on a transitive model.

**Exercise 6.** Give a translation that preserve *equivalence* of the basic temporal logic and prove it is correct. I.e. give a translation $ST$ from formulas of the basic temporal logic to formulas of first-order logic such that for all models $\mathcal{M}$, states $w \in \mathcal{M}$ and assignment $g$:

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}, g[x \mapsto w] \models ST_x(\varphi).$$

**Exercise 7.** The *difference* operator is defined as follows:

$$\mathcal{M}, w \models D\varphi \text{ iff there is } u \neq w \text{ s.t. } \mathcal{M}, u \models \varphi.$$

(a) Define the universal modality using $D$.

(b) Extend the standard translation to $D$.

**Exercise 8 (For Grad Students).** Modify the definition of $ST$ as follows

$$\begin{array}{rcl} ST_x(p_i, n) & = & P_i^n(x) \\ ST_x(\neg\varphi, n) & = & \neg ST_x(\varphi, n) \\ ST_x(\varphi \wedge \psi, n) & = & ST_x(\varphi \wedge \psi, n) \\ ST_x(\langle R_i \rangle \varphi, n) & = & \exists y(R_i^n(x, y) \wedge ST_y(\varphi, n+1)) \end{array}$$

Prove that the translation preserves *satisfiability* for BML over the class of all models. (Hint: Pay attention to the change in signature. Think unravelling.)