Worksheet

Hybrid Logics

Exercise 1.

1. Show two models that are bisimilar for the basic modal language but that can be distinguished by a formula of $\mathcal{H}(@)$.

2. Prove that formulas in $\mathcal{H}(@)$ are invariant under $\mathcal{H}(@)$-bisimulations. I.e., prove the following proposition.

   Proposition: Let $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2 \rangle$ be two hybrid models, and Z an $\mathcal{H}(@)$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $w_1 Z w_2$. Then for all formulas $\varphi \in \mathcal{H}(@)$ $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.

3. Prove that if the models are *image finite* (i.e., every element in the domain has a finite number of successors) then the converse is true.

   Proposition: Let $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2 \rangle$ be two image finite hybrid models. Furthermore, assume that for all formulas $\varphi \in \mathcal{H}(@)$ $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$. Then there is a $\mathcal{H}(@)$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $w_1 Z w_2$.

Exercise 2. Write in detail the proof of completeness for the tableaux for the BML presented in class.

Exercise 3. Extend the tableaux for the basic modal logic introduced in class with the following rule for the inverse modality $\Diamond^-$ with semantics: $\mathcal{M}, w \models \Diamond^- \varphi$ iff $\exists w'$ s.t. $R(w', w)$ & $\mathcal{M}, w' \models \varphi$.

$$(\Diamond^-)\ \frac{s{:}\Diamond^- \varphi}{\begin{array}{c} tRs \\ t{:}\varphi \\ t \text{ a new label} \end{array}} \qquad (\neg\Diamond^-)\ \frac{\begin{array}{c} s{:}\neg\Diamond^- \varphi \\ tRs \end{array}}{t{:}\neg\varphi}$$

Prove that the tableaux is sound and complete for the intended semantics.

Exercise 4. Change the $(\neg\Diamond)$ rule from the tableaux for the BML for the following one

$$(\neg\Diamond_4)\ \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{\begin{array}{c} t{:}\neg\varphi \\ t{:}\neg\Diamond\varphi \end{array}}$$

1. Prove that the resulting tableaux is sound and complete for the basic modal logic interpreted on the class of transitive models (i.e., K4).

2. Give an example of a formula where the run of the tableaux is non-terminating.

Exercise 5. Using the tableaux for $\mathcal{H}(@)$ introduced in class prove that the following hybrid formulas are valid on the class of all models.

1.	Self-dual	$@_i p \leftrightarrow \neg @_i \neg p$
2.	Nom	$@_i j \wedge @_j p \rightarrow @_i p$
3.	Introduction	$i \wedge p \rightarrow @_i p$
4.	Back	$\Diamond @_i p \rightarrow @_i p$
5.	Bridge	$\Diamond i \wedge @_i p \rightarrow \Diamond p$

Exercise 6. Redefine clash to be: "A branch Γ has a *clash* if $\{i{:}\theta, i{:}\neg\theta\} \subseteq \Gamma$ (i and θ arbitrary)". Using this prove that the following *formula schema* is valid in $\mathcal{H}(@)$:

$$i{:}j \to (i{:}\varphi \to j{:}\varphi).$$

Exercise 7. Using the tableaux for $\mathcal{H}(@)$ introduced in class find a model for the following hybrid formulas

$$\Diamond p \wedge \Diamond \neg p \wedge \Box(\Diamond p \wedge \Diamond \neg p) \wedge \Box\Box\Diamond i.$$

Note: Spell out exactly the model obtained from the tableaux (and not an arbitrary model for the mentioned formulas).

Exercise 8. Propose tableau rules for $\downarrow$. Prove that they are sound. Optional: prove completeness too.

Exercise 9. (For Grad Students) Extend the standard translation to $\mathcal{H}(@, \downarrow)$ and prove that it preserves equivalence of formulas.