Modal Logics: A Modern Perspective

Carlos Areces
carlos.areces@gmail.com

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Stanford

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  “Modal Logic: A Modern Perspective”

  an advanced course on Modal Logics.

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  - Good knowledge of Propositional Logic and First-Order Logic
  - (Some) Previous knowledge of Modal Logic

What we can cover

- **Unit I:** Introduction (Aims and Evaluation, Methodology). Recap of Propositional and First Order Logic. Basic Modal Logic (Syntax and Semantics). Motivation, Examples, Applications. Other Modal Operators.
- **Unit II:** Modal Logics as Fragments of Classical Logics. The Standard Translation. Transference of Results (Decidability via Translation). Optimized Translations. Automated Reasoning via Translations.
- **Unit III:** Hybrid Logics. Decidable Hybrid Logics (Tableau Calculi). The $\downarrow$ and @ operators and Undecidability. Axiomatizations (Henkin Models). Hybrid XPath (XPath as a query language, Axiomatization, Tableau Calculus).
- **Unit IV:** Description Logics. Web Ontologies and the Semantic Web. Knowledge Bases. A-Box and T-Box Reasoning (Tableau Calculi, Termination).
- **Unit V:** Dynamic Modal Logics. Epistemic Logics (Applications and Limitations). Dynamic Epistemic Logics (Public Announcement Logic, Action Modal Logics, Expressive Power, Decidability). Other Dynamic Modal Operators (Fragments of First Order Logic, Undecidability).

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  - First Steps in Modal Logic. Popkorn

Myself

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- Email: `carlos.areces@gmail.com`
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- Studied CS at Universidad de Buenos Aires, then I did a PhD degree in Logic at ILLC Amsterdam (supervisor Maarten de Rijke, promotor Johan van Benthem), then a postdoc there, then 7 years at INRIA-Nancy in France.

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- Now just arrived at Stanford University (and it is great!).

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Why “Logics”

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- How many logics are there?
- In this course we are going to
  - Discuss different modal logics, from different perspectives
  - Take, mainly, a model theoretic approach (e.g., bisimulations)
  - Discuss computational issues (e.g., complexity)

Simple structures for simple languages

Think of a colored graph:

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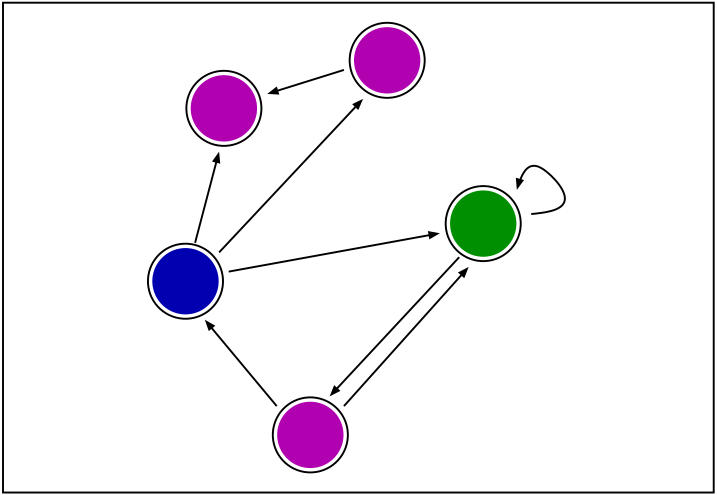

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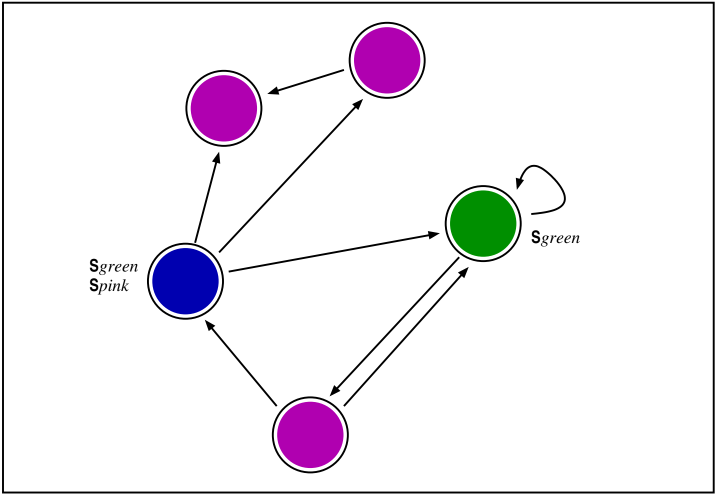

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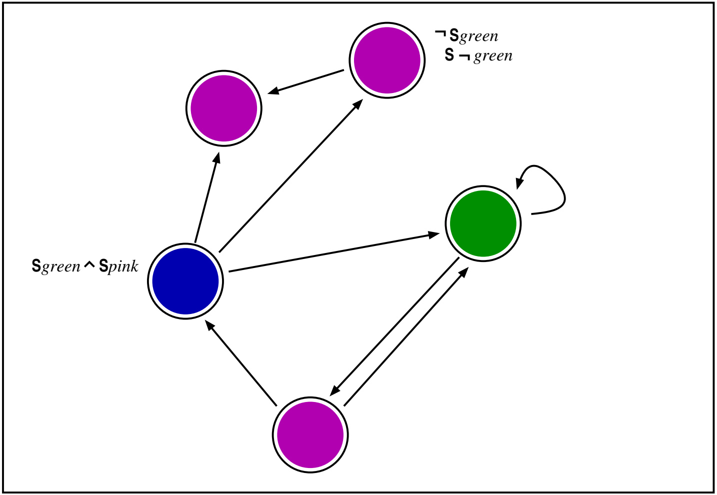

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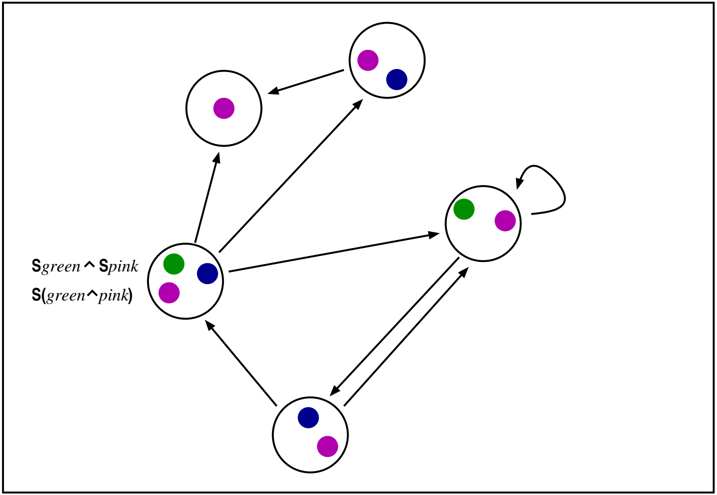

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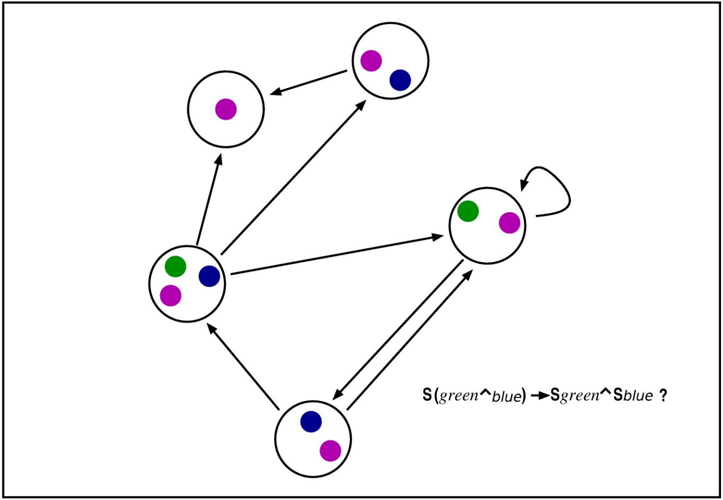

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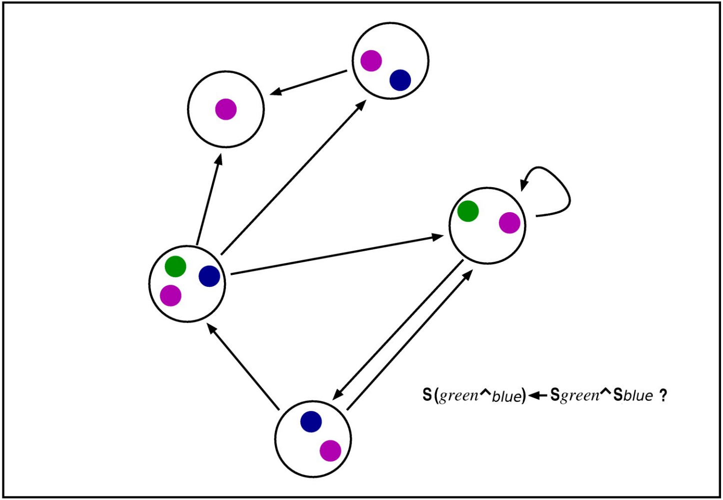

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- First advantage:
  - To decide weather a first-order formula is satisfiable, is undecidable.
  - In the modal logic we just presented, it is computable! (actually, PSPACE-complete)

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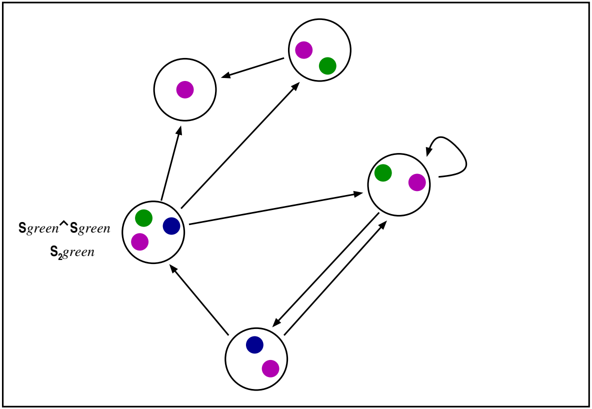

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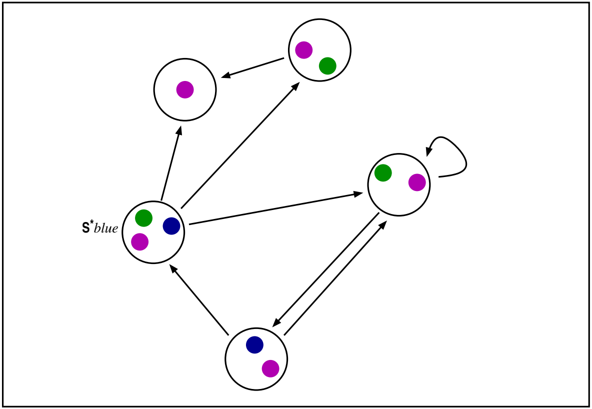

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- Some questions we can ask:
  - Which are the expressivity limits of these languages?
  - Can we define inference algorithm for these languages?
  - How efficient are they?
- An alternative perspective is to look at them from the perspective of logic engineering. Give a problem that requires inference:
  - Which is the best language we can use? (e,g. the easiest to use)
  - Which logic has good inference algorithms?
  - Which operators do I need?

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- **Why?** Because many things can be represented as colored graphs (i.e., relational structures).

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- Where "interesting means
  - Decidable, expressive, of "low" complexity, modular, etc.

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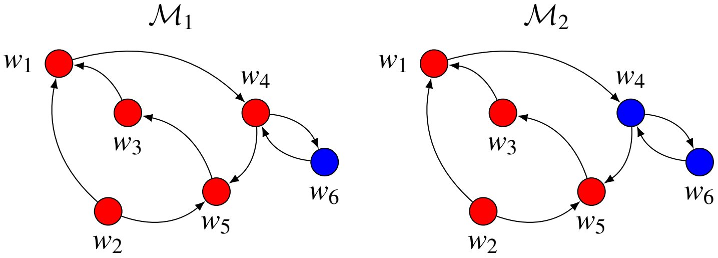
- Let φ be an arbitrary sentence, and $\mathcal{M}$ a first-order model (in the signature of φ). Then φ will either be true or false in (all) $\mathcal{M}$. We cannot talk of a particular part of the model with φ.



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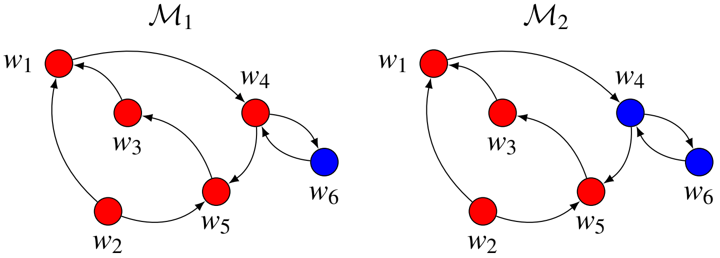

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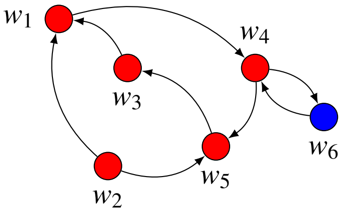
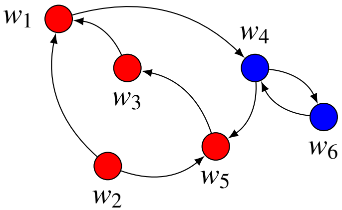


$\mathcal{M}_2$



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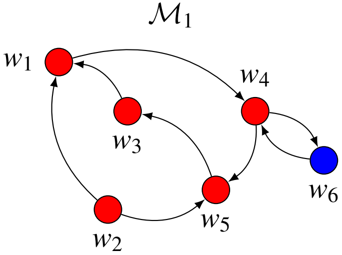
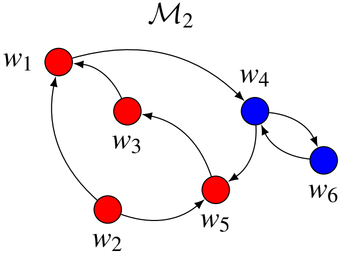
- We use formulas with free variables:





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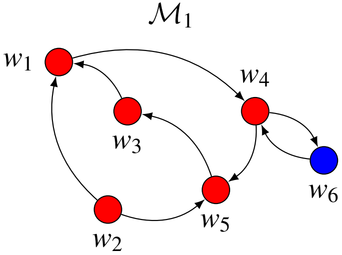

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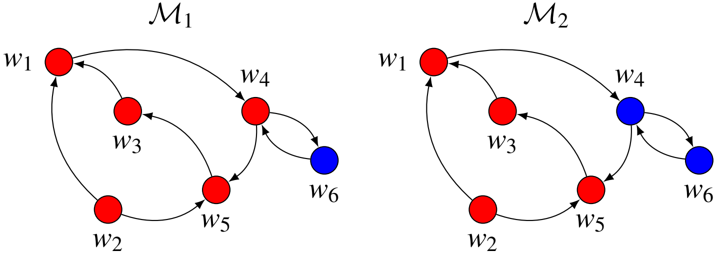

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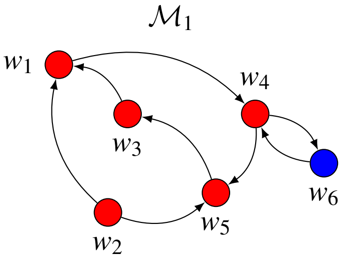

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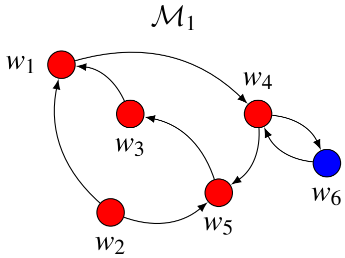

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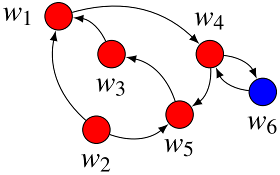
Internal Perspective

- Things are easier using a modal language:



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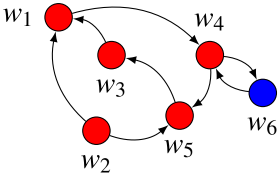

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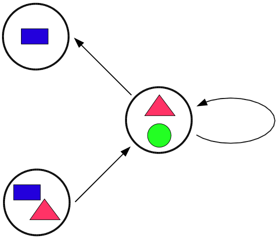

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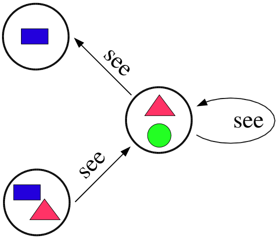

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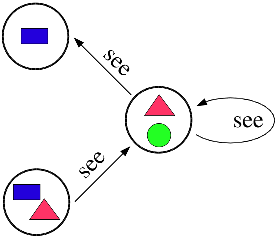

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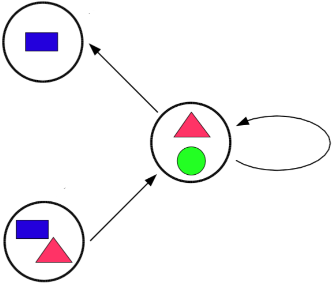
From the perspective of a node n, the meaning of the classical modal operators would be:

- $\langle see \rangle x$ = "n can see figure x in one neighbor.
- $[see]x$ = "n can see figure x in all neighbors.

Describing Labeled Graphs

Now we can "query" the model:



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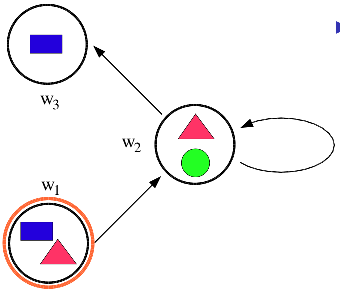

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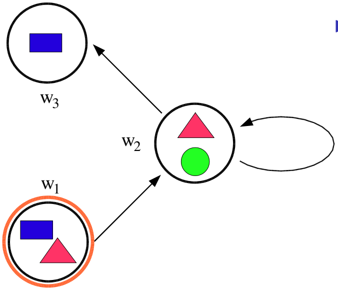

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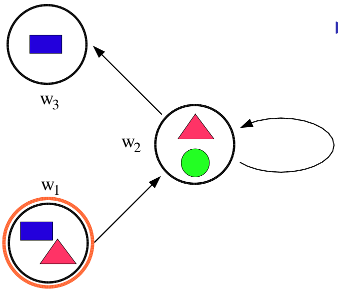

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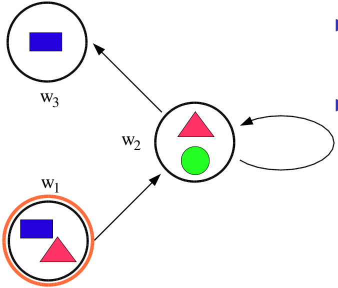

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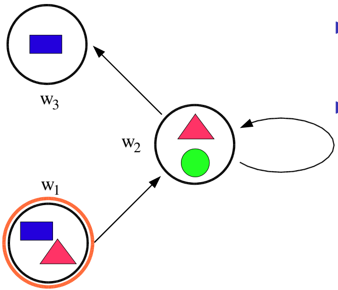

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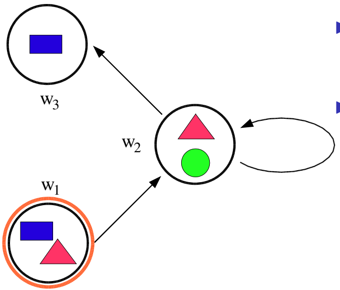

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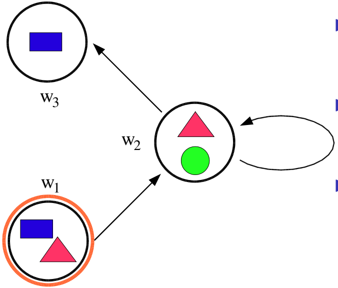

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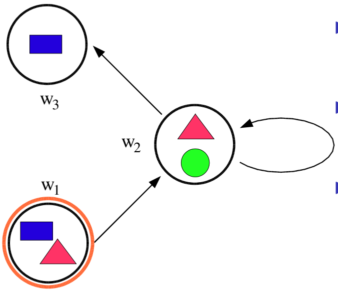

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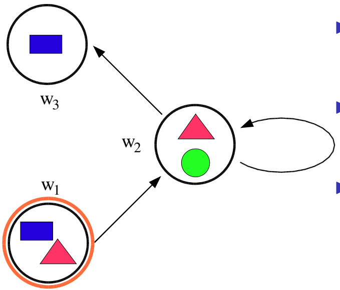

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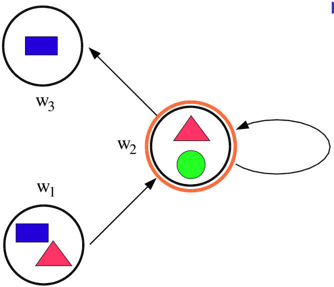

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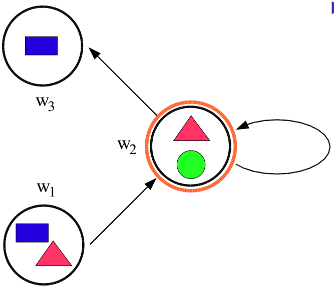

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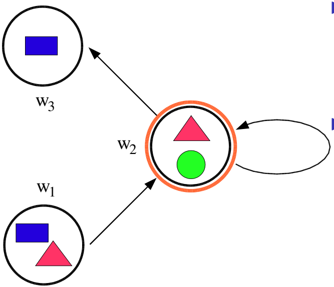

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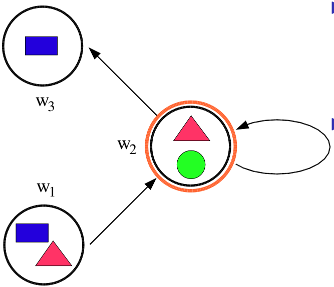

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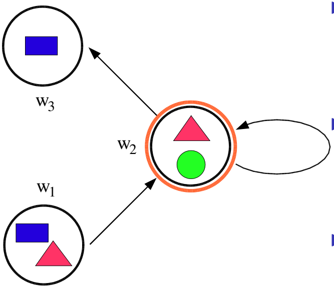

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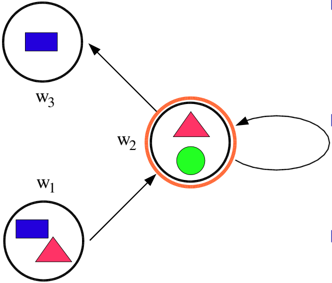

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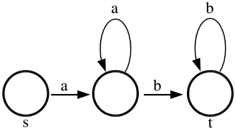

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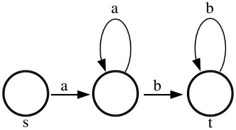

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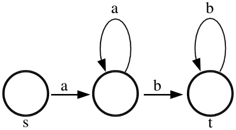

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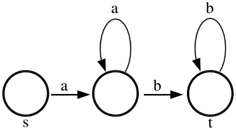

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- In this case we have to modal operators $\langle a \rangle$ and $\langle b \rangle$
- All formulas of the form

$$\langle a \rangle \ldots \langle a \rangle \langle b \rangle \ldots \langle b \rangle t$$

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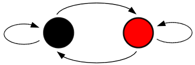

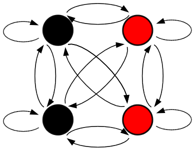

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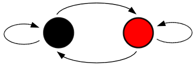

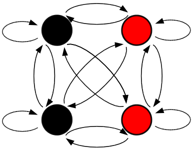

- FOL (with equality) can easily distinguish between these two structures (how?). The basic modal language does not.

The Basic Modal Language: Syntax and Semantics

- The syntax of the basic modal language is defined in terms of a signature defined by two infinite, countable, disjoint sets:
  - $\mathrm{PROP} = \{p_1, p_2, \ldots\}$, the set of *propositional variables*.
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- The operator $[m]$ is defined as $[m]\varphi = \neg\langle m \rangle\neg\varphi$ (similarly as how $\forall x.\varphi$ is defined as $\neg\exists x.\neg\varphi$).

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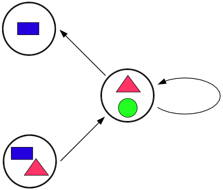
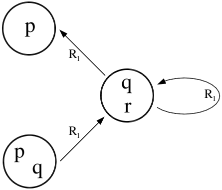
- Notice that $\mathcal{M}$ is a labeled directed graph or, from a more classical perspective, a relational structure.





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- We say that φ is valid in $\mathcal{M}$ iff for all $w \in W$ $\mathcal{M}, w \models \varphi$, and we write $\mathcal{M} \models \varphi$.

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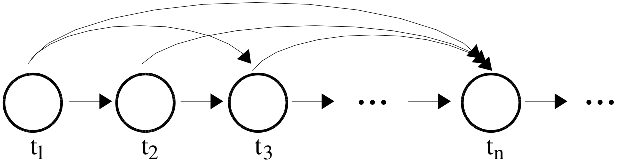

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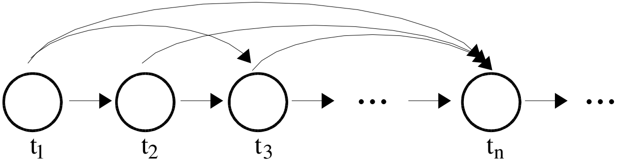

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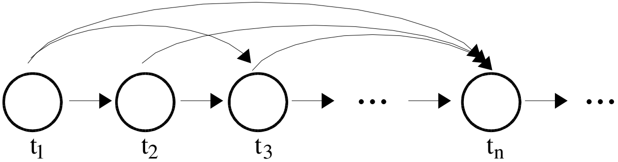

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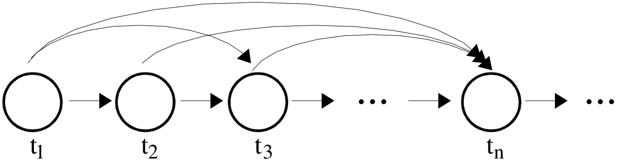

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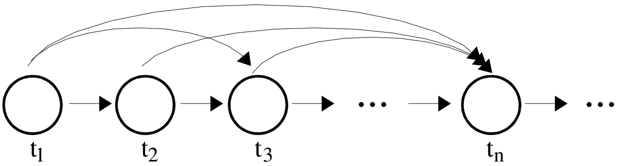

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- The $\langle R \rangle$ operator means in some point in the future.
- The $[R]$ operator means in every point in the future.
- Can we say, in this language, in some point in the past?

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- How can we define the semantics of the new operator?

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- What is the formal semantics for A?
- And a question to think about: is this operator the same as the $\forall$ operator in FOL?

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- Now we can model the inductive structure of progrmas: they can be composed, iterated, etc., building up other programs.

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E.g. if $\langle \pi_1 \rangle$ and $\langle \pi_2 \rangle$ are modal operators, then $\langle \pi_1 \cup \pi_2 \rangle$, $\langle \pi_1; \pi_2 \rangle$ and $\langle \pi^* \rangle$ are modal operators.

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- Then,

$$\mathcal{M}, w \models \langle \pi \rangle \varphi \text{ iff } \exists w'. R_\pi(w, w') \text{ y } \mathcal{M}, w' \models \varphi.$$

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- Think about induction...

Tests

Sometimes, it is useful to include tests as programs.

- Syntax: If φ is a PDL formula, then $\varphi?$ is a PDL program.
- Semantics: $R_{\varphi?} = \{(w, w) \mid \mathcal{M}, w \models \varphi\}$

Notice how we can now "transform" formulas into programs. What do you think of the following formula

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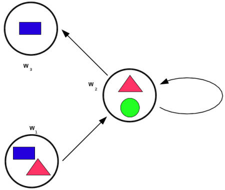
Are they useful then?

Extensions: Hybrid Logics

Let us go back to simple labeled graphs.



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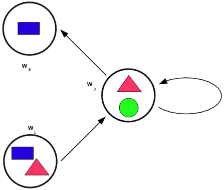

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- The basic modal language does not have constants or equality.
  We can add means to name and compare states.

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$\mathcal{HL}(@)$ is the extension of the basic modal logic with

- names (nominals): they are a news set of atomic symbols, used to represent states. The key is to ensure that a nominal is true at a single state in a model. In general we will write them as $i, j, k, \ldots$
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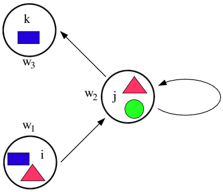

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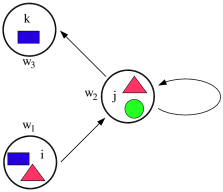

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- Exercise!