Hybrid Logics

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Stanford

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- Talk a bit about decidability / complexity.
- We will define a tableax algorithm for satisfiability of formulas in the basic modal logic.
  - How do we check whether a formula has a model?
  - What can we learn from tableaux?
- Transform the tableau for the basic modal logic into one for $\mathcal{H}(@)$.

Relevant Bibliography

- Blackburn, P., de Rijke, M. and Venema, Y. Chapter 7, Section 3 of “Modal Logic”, Cambridge Tracts in Theoretical Computer Science, 53, Cambridge University Press, 2001.

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Syntax: Propositional Logic + modalities

Semantics: Interpreted in terms of relational structures (Graphs)

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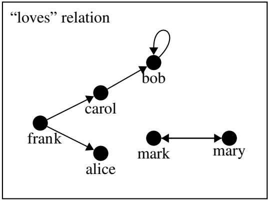

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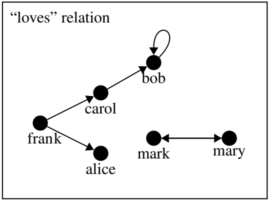

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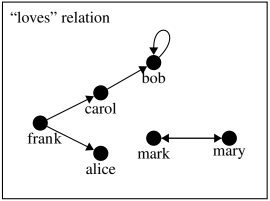

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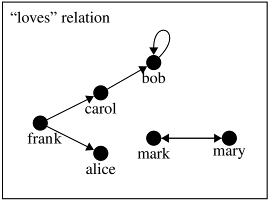

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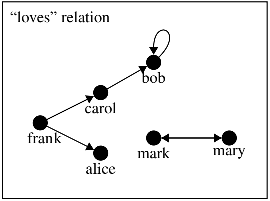
- MLs can be (usually) thought of as fragments of FO
- Modal logics are (usually) decidable
  - SAT for the basic modal logic is PSpace-complete

The Limits of Modal Expressivity

Some properties can't be expressed in the basic modal language…



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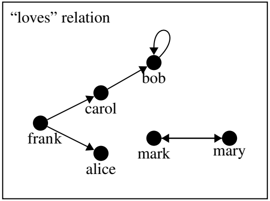

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- What do we need?
  - constants
  - identity

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  - A representation problem: for some applications modal logic is not adequate as a representation formalism, and
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  - A representation problem: for some applications modal logic is not adequate as a representation formalism, and
  - A reasoning problem: modal reasoning systems are difficult to devise.
- These limitations motivated the work on Hybrid Logics.

The Basic Recipe: $\mathcal{H}(@)$

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- Nominals denote elements (nodes) in the model
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- Nominals denote elements (nodes) in the model
- $@_i\varphi$ is true iff φ is true in the element denoted by i.
  In particular $@_ij$ says that i and j denote the same point in the model (i.e., $i = j$).

The Hybrid Logic $\mathcal{H}(@)$

Syntax:

$$\text{FORM} := p \mid i \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \langle R\rangle\varphi \mid [R]\varphi \mid @_i\varphi,\ i \in \text{NOM}$$

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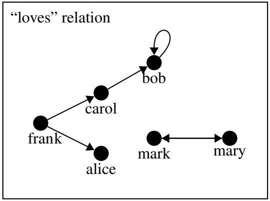
Semantics: Restrict valuation V so that $V(i)$ is a singleton for $i \in \mathsf{NOM}$.

We define
$$\begin{array}{rcl} \mathcal{M}, w \models i & \text{iff} & w \in V(i) \text{ (iff } V(i) = \{w\}) \\ \mathcal{M}, w \models @_i\varphi & \text{iff} & \mathcal{M}, w' \models \varphi \text{ for } w' \in V(i) \end{array}$$

The Expressive Power of $\mathcal{H}(@)$



Query: "Are there two people who loves each other?"

In $\mathcal{H}(@)$: $\varphi := @_{mark}\langle loves\rangle mary \wedge @_{mary}\langle loves\rangle mark \wedge @_{mary}\neg mark$

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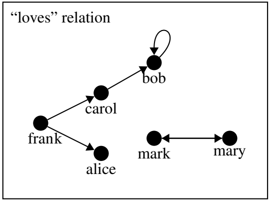

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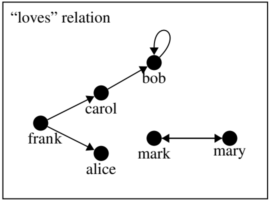

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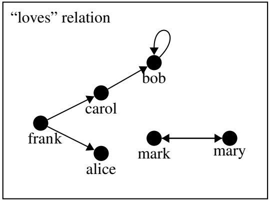

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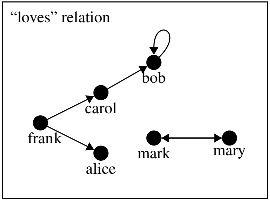

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Definition. A bisimulation for BML is a non empty binary relation Z between the domains of two models $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2 \rangle$ s.t.

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- Consider BML extended with @. Do we need any change? Yes, $@_i$ moves evaluation to the point named by i, that might not be reachable by the Zig/Zag conditions.
- Bisimulation for $\mathcal{H}(@)$: To the conditions for BML-bisimulation add the followng conditions:
  - For all $i \in \mathsf{NOM}$, $i^{\mathcal{M}_1} Z i^{\mathcal{M}_2}$.

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- Think about it for next class.

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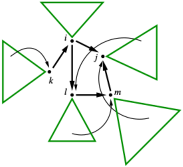

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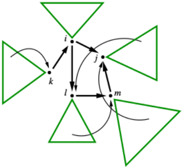

  - (The hybrid μ-calculus with past and the universal modality is ExpTime-complete, and the proof uses tree-automata).

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- We say that a problem P is decidable, if we have an algorithm that given any instance of P terminates after a finite number of steps answering correctly correctly YES or NO.
- Once we know that a problem is decidable, we can investigate how expensive it is to solve it. We usually look at two parameters:
  - Time: How many steps takes algorithm A to find a solution?
  - Space: How much memory uses algorithm A to find a solution?

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- All these choices gives as many complexity classes, like:

$$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSpace} \subseteq \mathrm{NPSpace} \subseteq \mathrm{Exp} \subseteq \mathrm{NExp} \subseteq \mathrm{ExpSpace} \subseteq \mathrm{NExpSpace}$$

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- What about the basic modal language?

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- They will help us keep track of what is going on in each point in our model.

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- It is actually terminating, and by imposing some restrictions on applications it can run in PSPACE, so it is optimal.
- To think: what is the status of labeled formulas $s{:}\varphi$ and accessibility statements sRt?

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s{:}(\Diamond p \wedge (\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q))) \\
s{:}\Diamond p \\
s{:}\neg\Diamond\neg q \wedge \neg\Diamond(p \wedge q) \\
s{:}\neg\Diamond\neg q \\
s{:}\neg\Diamond(p \wedge q) \\
sRt \\
t{:}p \\
t{:}\neg\neg q \\
t{:}q \\
t{:}\neg(p \wedge q) \\
t{:}\neg p \qquad\qquad t{:}\neg q
\end{array}$$

closed closed

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}} \qquad \frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

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t{:}p \\
t{:}\neg\neg q \\
t{:}q \\
t{:}\neg(p \wedge q) \\
t{:}\neg p \qquad\qquad t{:}\neg q \\
\\
\text{closed} \qquad\qquad \text{closed}
\end{array}$$

- Which are the similarities/differences with tableaux for PL?

The Complete Cast, plus an Example

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

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$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

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t{:}\neg p \qquad\qquad t{:}\neg q \\
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\text{closed} \qquad\qquad \text{closed}
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- Which are the similarities/differences with tableaux for PL?
- How do we know that we got it right?

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- Which are the similarities/differences with tableaux for PL?
- How do we know that we got it right?
- What can we learn from the calculus?

A Closer Look

A Closer Look

- Which similarities / differences with tableaux for PL?

$$\frac{s{:}(\varphi \land \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

$$\frac{s{:}\neg(\varphi \land \psi)}{s{:}\neg\varphi \qquad\qquad s{:}\neg\psi}$$

$$\frac{s{:}\neg\neg\varphi}{s{:}\varphi}$$

$$\frac{s{:}\Diamond\varphi}{\begin{array}{c} sRt \\ t{:}\varphi \end{array}}$$

for t a new label

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?

$$\frac{s{:}(\varphi \wedge \psi)}{\begin{array}{c} s{:}\varphi \\ s{:}\psi \end{array}}$$

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$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \qquad\qquad s{:}\neg\psi}$$

$$\frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

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A Closer Look

- Which similarities / differences with tableaux for PL?
  - What are labels? What are they doing? Can we use them?
  - Is this an algorithm?

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$$\frac{s{:}\neg(\varphi \wedge \psi)}{s{:}\neg\varphi \quad\mid\quad s{:}\neg\psi} \qquad \frac{\begin{array}{c} s{:}\neg\Diamond\varphi \\ sRt \end{array}}{t{:}\neg\varphi}$$

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- Which similarities / differences with tableaux for PL?
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- Did we get it right?

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- Did we get it right?
  - Did we get it right in the PL case, to start with?! Consider the rule:

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- What can we learn from the calculus?

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  - We should prove soundness and completeness
- What can we learn from the calculus?
  - Something about models!

Tree Models

Tree Models

- Let us see the tableux proof we did before again, for the formula

$$\varphi = s{:}(\langle R\rangle p \wedge (\neg\langle R\rangle\neg q \wedge \neg\langle R\rangle(p \wedge q)))$$

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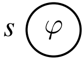

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$$s{:}\neg\langle R\rangle\neg q \wedge \neg\langle R\rangle(p \wedge q)$$

s

$$\langle R\rangle p$$
$$\neg\langle R\rangle\neg q \wedge \neg\langle R\rangle(p \wedge q)$$

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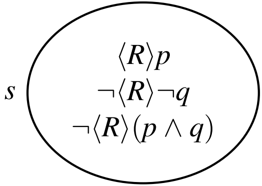

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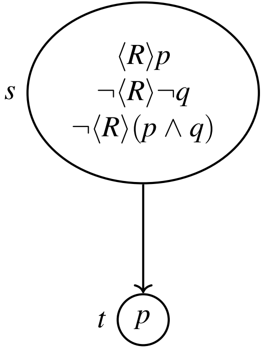

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$$t{:}p$$

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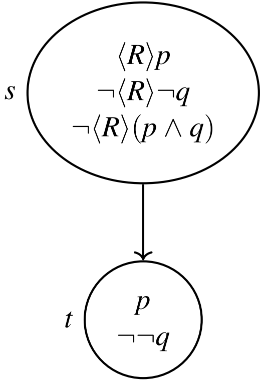

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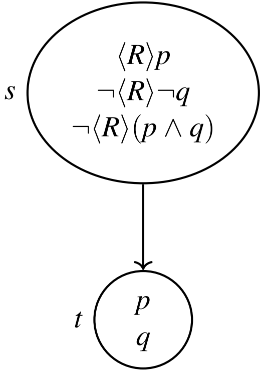

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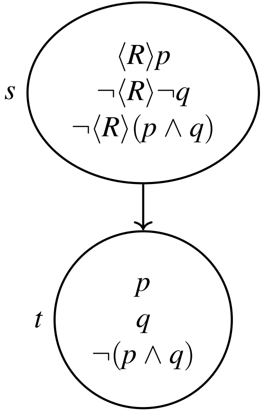

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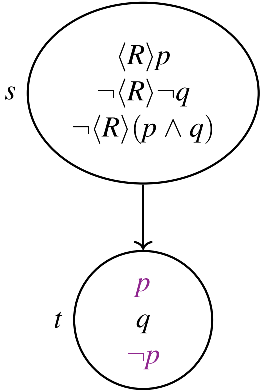

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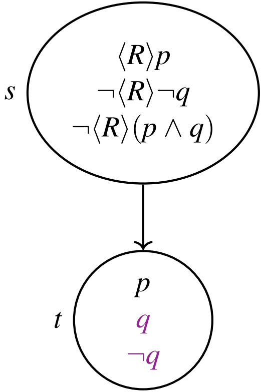

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Tree and Finite Model Properties

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- Using the rules of the tableaux calculus we only explore finite, tree models.

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  - cannot say infinite,
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Theorem: A formula in the $\langle R \rangle$-language is satisfialle if and only if it is satisfiable in a finite, tree relational structure.

Soundness and Completeness

- We would like to verify that:
  - if the tableau for φ is closed (all tranches contains a class) then φ is UNSAT [Soundness].
  - if the tableau for φ cannot be further extended and it has an open branch then φ is SAT [Completeness].

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- Soundness is usually easy to establish. Prove, for each rule of the tableaux, that if the antecedent has a model, then at least one of the generated branches has a model.
- To show completeness we need to build a model from a saturated, open branch.

Completeness

Theorem. If Γ is a saturated open branch from a tableaux for φ, then φ is SAT.

Proof. Given Γ we define the model $\mathcal{M}_\Gamma = \langle W_\Gamma, R_\Gamma, V_\Gamma \rangle$ where

$$\begin{array}{rcl} W_\Gamma & = & \{w \mid w{:}\varphi \in \Gamma\} \\ R_\Gamma & = & \{(w,v) \mid wRv \in \Gamma\} \\ V_\Gamma(p) & = & \{w \mid w{:}p \in \Gamma\} \end{array}$$

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Let ψ be the smallest formula such that $w{:}\psi \in \Gamma$ and $\mathcal{M}_\Gamma, w \not\models \psi$.

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Hence, $w{:}\varphi \in \Gamma$ implies $\mathcal{M}_\Gamma, w \models \varphi$.

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- Fact 2: : is self-dual, hence $\neg i{:}\varphi$ is equivalent to $i{:}\neg\varphi$.

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Equality: $(\text{Ref})\ \dfrac{[i \text{ on branch}]}{i{:}i}$ $\qquad (\text{Sym})\ \dfrac{i{:}j}{j{:}i}$ $\qquad (\text{Cong})\ \dfrac{i{:}k \quad j{:}k \quad i{:}\varphi}{j{:}\varphi}$

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Bolander, T. and Blackburn, P..
Termination for Hybrid Tableaus
Journal of Logic and Computation, 17, 517-554, 2007.

A Tableau Based Prover for Hybrid Logics

- A sound, complete and terminating calculus has been implemented in the prover HTab
- Available at `https://hackage.haskell.org/package/HTab`.
- Implemented in haskell, with a GPL license. I.e., you can get the code and change it!
- Demo