Hybrid Logics

Completeness

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Bibliography

- Modal Logic Book, Chapter 4, by Blackburn, Venema and de Rijke.

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- The question of when these two alternatives define the same logic arises.
- More precisely, we want to see if the set of valid formulas is exactly the same as the set of theorems.
- The answer is found by results of correctness and completeness that show whether $\models$ and $\vdash$ are equal.

Modal Logics are Old

- Aristotle (384BC – 322BC) was already a modal logician.
- In addition to "classical syllogism," Aristotle discusses modal syllogism that results from adding the qualifications "necessarily" and "possibly" to premises and conclusions, in various ways.
- He actually already discusses different modal logics, as he considers two possibly definitions of "possibly P":
  - "possibly P" as equivalent to "not necessarily not P".
  - "possibly P" as equivalent to "not necessarily P and not necessarily not P".

Rini, Adriane.
Aristotle's Modal Proofs: Prior Analytics
A8–22 in *Predicate Logic*, Dordrecht: Springer, 2011.

Modal Logics are Simple

- Take the language of Propositional Logic and add the modal operators (this is the basic modal language $\mathcal{K}$)

$$\varphi, \psi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi$$

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- Classically
    - $\Diamond\varphi$ stands for "φ is possible"
    - $\Box\varphi$ stands for "φ is necessary"
- Now, argue away!
    - Should $\Box\varphi \to \varphi$ be valid?
    - What about $\Box\varphi \to \Box\Box\varphi$?
    - ...

Modal Logics are Flexible

- "Possibility" and "Necessity" are just two of the many possible options ...
  - Deontic Logic

$O\varphi$	It is obligatory that φ
$P\varphi$	It is permitted that φ
$F\varphi$	It is forbidden that φ

  - Temporal Logic

$G\varphi$	It will always be the case that φ
$F\varphi$	It will be the case that φ
$H\varphi$	It has always been the case that φ
$P\varphi$	It has been the case that φ

  - Doxastic Logic

$B_x\varphi$	x believes that φ

  - ...

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With this semantics, both $\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$ are valid.

The Accessibility Relation

By introducing an accessibility relation between possible worlds it was possible to define weaker logics.

Nowadays, a Kripke model is a structure $\mathcal{M} = \langle W, R, V \rangle$ where W is a non-empty set, $R \subseteq W^2$ and $V : W \to 2^{\mathsf{PROP}}$.

- I.e., A directed, labeled graph,
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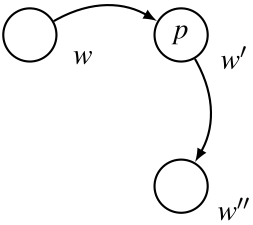

Examples

- $\mathcal{M}, w \not\models \Box p \to p$
- $\mathcal{M}, w \not\models \Box p \to \Box\Box p$

Completeness

- It was discovered that some axioms correspond to properties of the accessibility relation
  - $\Box p \rightarrow p \leftrightsquigarrow R$ is reflexive
  - $\Box p \rightarrow \Box\Box p \leftrightsquigarrow R$ is transitive
- ... and the first completeness results were proved.

Kripke, Saul.
A completeness theorem in modal logic.
The Journal of Symbolic Logic, 24, 1959.

Kripke, Saul.
Semantical analysis of modal logic I. Normal modal propositional calculi.
Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 9:67–96, 1963.

More Completeness

- Kripke's completeness proof employed a generalization of Beth's tableaux method.
- Completeness was established by showing how to derive a proof from a failed attempt to find a counter model.

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- Completeness was established by showing how to derive a proof from a failed attempt to find a counter model.
- Kaplan (1966) criticized Kripke's proof as lacking in rigor and as making excessive use of "intuitive" arguments on the geometry of tableau proofs.
- He suggested a different, arguably, more elegant approach based on an adaptation of Henkin's model theoretic completeness proof for first-order logic.
  (Kaplan was not the first neither the only: Bayart 1959, Makinson 1966, Cresswell 1967)

Henkin's Completeness

- Henkin's completeness proof for first-order logic uses (at least) two important ideas
  1. A consistent set of formulas can be extended to a maximally consistent set of formulas
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- (1) is key in modern completeness proofs for propositional modal logics which build a canonical model (satisfying all consistent formulas) that has as domain the (uncountable) set of all maximally consistent sets of formulas.
- (2) seemed less useful in a propositional setting.

Basic Definitions

- A Modal Logic Δ is a set of modal formulas that contain all propositional tautologies and is closed under:
  i. Modus Ponens: If $\varphi \in \Delta$ y $\varphi \rightarrow \psi \in \Delta$ then $\psi \in \Delta$.
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- If $\varphi \in \Delta$ we say that φ is a theorem of Δ, and we write $\vdash_\Delta \varphi$.
- If Δ_1 and Δ_2 are modal logics such that $\Delta_1 \subseteq \Delta_2$ we say that Δ_2 is an extension of Δ_1

Some Examples

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- For example, the logic generated by the empty set contains all the instances of propositional tautologies and nothing else. We call it PC.

Syntactic Consequence

- If $\Gamma \cup \{\varphi\}$ is a set of formulas, then φ is deductible in Δ starting from Γ (notation $\Gamma \vdash_{\Delta} \varphi$) if there are formulas $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\Delta} (\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$.

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- A set of formulas Γ is Δ-consistent if $\Gamma \nvdash_\Delta \bot$. A formula φ is Δ-consistent if $\{\varphi\}$ is.

Normal modal logic

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- Now we will see a concept that is exclusively modal: normal modal logics.
- A modal logic Δ is normal if it contains the formula:

$$\textbf{(K)} \quad \Box(p \to q) \to (\Box p \to \Box q).$$

and is closed under the necesitation (or generalization) rule:

$$\textbf{(Nec)}\ \text{If } \vdash_\Delta \varphi \text{ then } \vdash_\Delta \Box\varphi.$$

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- The normal modal logic generated by the empty set is called **K**, and is the minimal normal modal logic. If Γ is a set of formulas, the normal modal logic generated by Γ is called $\mathrm{K}\Gamma$.

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Also, it is usual to say that Γ are the axioms. And to say that the logic is generated from Γ using the inference rules modus ponens, uniform substitution and generalization.

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The notion of local consequence is the strongest (i.e., $\Gamma \models^l_S \varphi$ implies $\Gamma \models^g_S \varphi$). Exercise.
From now on, $\Gamma \models_S \varphi$ es $\Gamma \models^l_S \varphi$.

Soundness & Completeness

Let's recap the definitions:

- A logic Δ is correct with respect to a class of models S if for all formula φ and for all model $\mathcal{M} \in S$, if $\vdash_\Delta \varphi$ then $\mathcal{M} \models \varphi$.

Soundness & Completeness

Let's recap the definitions:

- A logic Δ is correct with respect to a class of models S if for all formula φ and for all model $\mathcal{M} \in S$, if $\vdash_\Delta \varphi$ then $\mathcal{M} \models \varphi$.
- A logic Δ is strongly complete with respect to a class of models S if for any set of formulas $\Gamma \cup \{\varphi\}$, if $\Gamma \models_s \varphi$ then $\Gamma \vdash_\Delta \varphi$.

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That is:

- A normal modal logic Δ is strongly complete with respect to a class S iff each set of Δ-consistent formulas is satisfiable in some model $\mathcal{M} \in S$.

Canonical model

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- And the "pieces" that we will use are maximally consistent sets.

A set of formulas Γ is maximally Δ-consistent if Γ is Δ-consistent and any set of formulas that properly contains Γ is Δ-inconsistent.

- If Γ is a maximally Δ-consistent set we say that it is an Δ-MCS.

Canonical Model

What is the intuition behind using MCSs in a completeness proof for modal logics?

- Observe that each point w in each model $\mathcal{M}$ for a logic Δ is associated with a set of formulas $\{\varphi \mid \mathcal{M}, w \models \varphi\}$.

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- It is not difficult to see that such set of formulas is actually a Δ-MCS. And this means that if φ is true in a model for a logic Δ, then φ belongs to a Δ-MCS.
- Also, if w is related with w' in some model $\mathcal{M}$, then the information in each of the MCS associated to w and w' needs to be "coherently related".

Canonical Model

The idea is to work with collections of MCSs that are coherently related in order to construct the model we are looking for. The goal is to prove the truth lemma, that tell as that 'φ belongs to a MCS' is equivalent to 'φ is true in a model'.

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- The worlds of the canonical model are all the MCSs of the logic we are working with.
- We will see what it means for the information in the MCSs to be 'coherently related', and we will use this notion in order to define the accessibility relation.

Properties of MCSs

Let's begin by seeing some properties of the MCSs. If Δ is a logic and Γ is a Δ-MCS then:

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- $\Delta \subseteq \Gamma$.
- Γ is closed under modus ponens.
- For all formulas φ, $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
- For all formulas φ, ψ, $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Lindenbaum Lemma

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i) We numerate the formulas in our language $\varphi_1, \varphi_2, \ldots$.

ii) We define the sequence of sets:

$$\begin{array}{lcl} \Sigma_0 & = & \Sigma \\ \Sigma_{n+1} & = & \begin{cases} \Sigma_n \cup \{\varphi_n\} & \text{if the set is } \Delta\text{-consistent} \\ \Sigma_n \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases} \\ \Sigma^+ & = & \bigcup_{n \geq 0} \Sigma_n \end{array}$$

Construction of the canonical model

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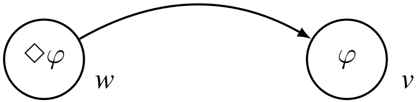

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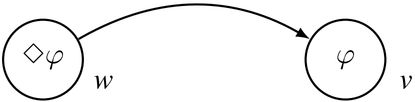

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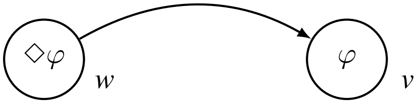

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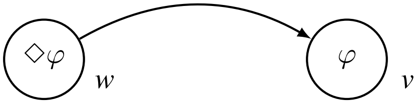

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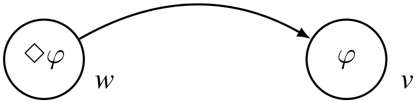

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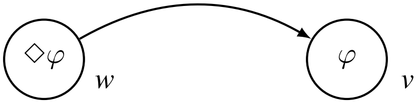

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  - The canonical valuation sets truth of a propositional symbol in w to membership in w.

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Proof: Suppose that $\Diamond\varphi \in w$. We will construct v such that $R^\Delta wv$ y $\varphi \in v$. Let $v^- = \{\varphi\} \cup \{\psi \mid \Box\psi \in w\}$.

- v^- is consistent. Exercise!

Then for Lindenbaum there exists a Δ-MCS v that extends v^-. By construction $\varphi \in v$, and for all formula ψ, $\Box\psi \in w$ implies $\psi \in v$.

- This last step implies that $R^\Delta wv$. Exercise!

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Now we can take "truth = membership" to the level of arbitrary formulas.

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Finally:

- **Theorem of the Canonical Model**: Any normal modal logic is strongly complete with respect to its canonical model.

Proof: Suppose that Σ is a Δ-consistent set. By the Lindenbaum lemma there exists a Δ-MCS Σ^+ that extends Σ. By the truth lemma, $\mathcal{M}^\Delta, \Sigma^+ \models \Sigma$.

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Corollary:

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- The normal modal logic **K** is strongly complete with respect to the class of all the models.

By the previous theorem, given that **K** is a normal modal logic, it is strongly complete with respect to its model $\mathcal{M}^K$. We just need to check that $\mathcal{M}^K$ belongs to the class of all the models, but this is trivial.

Hybrid Logic Completeness

- Remember the hybrid operators i (nominals) and $@$ (at).
- We will prove that by adding these operators to the language of the classical modal logic we can prove a general result of completeness.

Pure Formulas

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symmetry	$i \to \Box\Diamond i$	asymmetry	$i \to \neg\Diamond\Diamond i$
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density	$\Diamond i \to \Diamond\Diamond i$	universality	$\Diamond i$
determinism	$\Diamond i \to \Box i$	trichotomy	$@_j\Diamond i \vee @_j i \vee @_i\Diamond j$
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- **Claim:** When a pure formula is valid in a frame, it defines a first order property over its accessibility relation (Exercise).

Axiomatization of Pure Formulas

- When a pure formula is used as an axiom, the axiomatization is automatically complete over the class of frames defined by that formula
  Theorem: If P is the hybrid normal logic obtained by adding a set Π of pure formulas over the axiomatization of the minimal hybrid logic $\mathbf{K_h} + R$ (that we will introduce in a minute), then P is complete respect the class of frames defined by Π.

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- We will prove this general completeness using a constructon proposed by Henkin for first order logic.

From Valid in a Model to Valid in a Frame

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- If φ is a pure formula, we say that ψ is a pure instance of φ if it is obtained from φ substituting nominals by nominals.
- **Proposition:** Let $\mathcal{M} = \langle W, R, V \rangle$ be a named model and φ a pure formula. Suppose that for every pure instance ψ of φ we have that $\mathcal{M} \models \psi$. Then $\langle W, R \rangle \models \varphi$.

Normal Hybrid Logics

Δ is a normal hybrid logic if it is a normal modal logic and in addition it includes the following axioms and is closed under the following rules:

Axioms:

$K_{@}$	$@_i(p \to q) \to (@_i p \to @_i q)$
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Rules:

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- The proof of completeness also uses MCS. But in this case we use named MCSs. We say that a is named Γ by a nominal i if $i \in \Gamma$.

Properties of hybrid MCS

Proposition: Let Γ be a $\mathbf{K_h}$-MCS. For each nominal i we define $\Delta_i = \{\varphi \mid @_i\varphi \in \Gamma\}$.

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Maximality and the other points are exercises.

Canonical Model

At this point we could continue as we did for the case of **K**.

- Use the Lindenbaum lemma to show that every consistent set Σ can be extended as a maximally consistent set Σ^+.
- Consider the model whose domain is the set of all the MCS. (In fact we have to consider the submodel of the canonical model generated by $\Sigma^+ \cup \{\Delta_i \mid \Delta_i = \{\varphi \mid @_i\varphi \in \Sigma^+\}\}$, in order to make sure that we obtain a hybrid model).
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But the model obtained in this way is not a named model, and then if a pure formula is valid in this model does not imply that it is valid in its frame.

Admissible Rules

- As we mentioned, we can prove that the logic $\mathbf{K_h}$ is complete in a similar way as we did with K.
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- In order to construct a named model, we need two aditional rules.

Name	$\vdash j \to \theta$ then $\vdash \theta$
Paste	$\vdash @_i \Diamond j \wedge @_j \varphi \to \theta$ then $\vdash @_i \Diamond \varphi \to \theta$

  j is a nominal different from i which does not appear in θ.
- Let $\mathbf{K_h} + R$ be the logic obtained by adding these two rules to $\mathbf{K_h}$.

Named and Pasted

Let Δ be a normal hybrid logics, a Δ-MCS Σ is called

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Each Δ-MCS is a Full Model Description

Let Δ be a normal hybrid logic, then every Δ-MCS give rise to a collection of Δ-MCSs.

Lemma (Gosip Lemma)

Let Σ be a Δ-MCS. For any nominal $i \in \Sigma$, define $\Gamma_i = \{\varphi \mid @_i\varphi \in \Sigma\}$. Then Γ_i is a Δ-MCS that contains i.

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The Hybrid Canonical Model: Let Σ be a named and pasted Δ-MCS. Define $\mathcal{M}^c = \langle W^c, R^c, V^c \rangle$ where R^c and V^c are defined as before, and

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Facts

- $\mathcal{M}^c$ depends on Σ (we should actually talk about $\mathcal{M}^c_\Sigma$).
- $\mathcal{M}^c$ is countable.
- $\mathcal{M}^c$ is named: each state in W^c makes (at least) a nominal true.

Hybrid/Henkin Completeness

- The Truth and Existence Lemmas can be proved for $\mathcal{M}^c$ in a similar way as before.
- As a result, every normal hybrid logic is complete w.r.t. the class of models obtained from the set of its MCS.

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And now for the magic. Remember that:

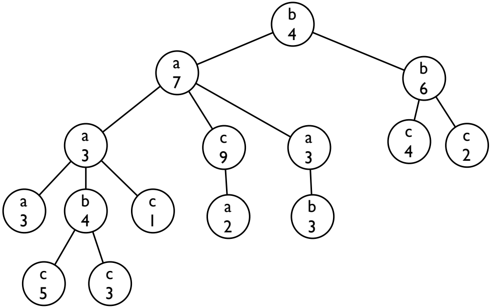
- Fact: Let $\mathcal{M} = \langle W, R, V \rangle$ be named and φ pure. If for all pure instances ψ of φ, $\mathcal{M} \models \psi$, then for all V', $\langle W, R, V' \rangle \models \varphi$.
- Fact: If φ is pure and for all V', $\langle W, R, V' \rangle \models \varphi$ then φ defines a first-order property on R.

- Hence, if $\mathbf{K_h} + R$ is extended by a set of pure axioms Π then the resulting logic is automatically strongly complete w.r.t. the class of models defined by Π.

An Application to Semi-Structured Databases

- Many data-intensive applications use complex data that is not naturally encoded in a relational database.
  - Web data or biological data are better described using semi-structured data models, that organize information as labeled trees or graphs.
  - They contain labels from a finite alphabet (the structural information), and from an infinite alphabet (the actual data).
- A well known example is XML (eXtensible Markup Language), the most successful data model of this kind.
- XPath is the most widely used XML query language.

XPath with data tests



Data tree

- ordered tree (finite or infinite)
- nodes have
  - a label (finite alphabet)
  - a data (infinite domain)

Path expressions

- go to ancestor, check *a*, go to child, check *c*, go right sibling, go to descendant, check *b*

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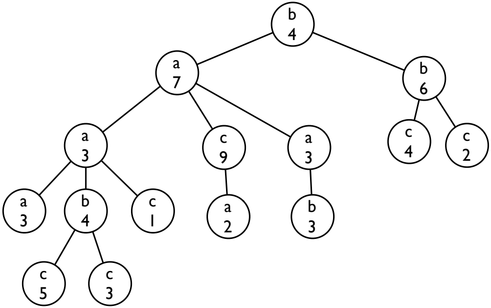

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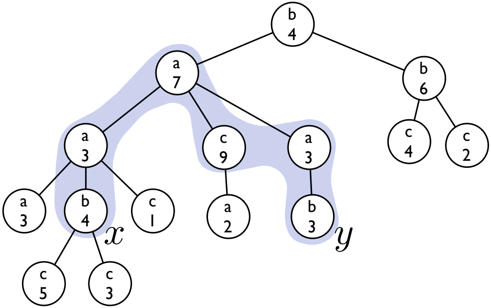

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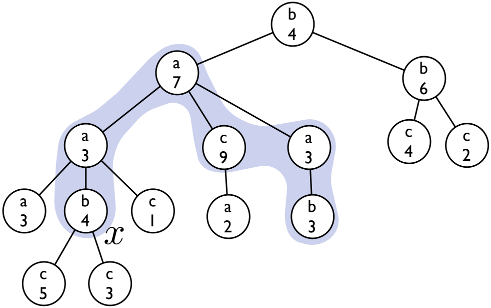

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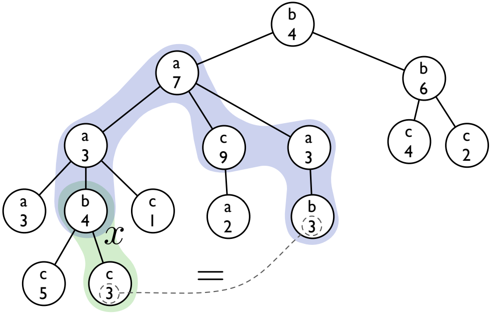

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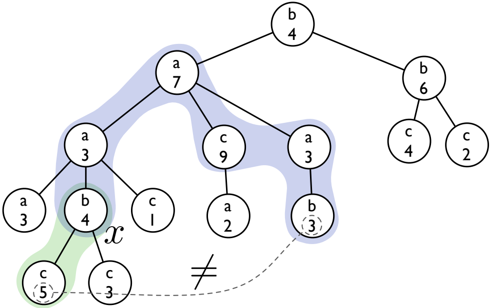

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$\langle @_i = @_j \rangle$	The node named i has the same data than the node named j.
$\langle \alpha = @_i\beta \rangle$	There is a node accessible from the current point by an α path that has the same data than a node accessible from the point named i by a β path.

Axiomatizating XPath

- An axiomatization for $\text{XPath}_=(\downarrow)$ was first given by Abriola et al. but the completeness proof is non trivial
- We then proposed $\text{HXPath}_=(\downarrow\uparrow)$ where a Henkin completeness proof was possible.

Abriola, S., Descotte, M. , Fervari, R., and Figueira, S.
Axiomatizations for downward XPath on Data Trees.
Journal of Computer and System Sciences, 2017.

Areces, C. and Fervari, R.
Hilbert-style Axiomatization for Hybrid XPath with Data.
In *Proceedings of the 15th European Conference On Logics In Artificial Intelligence*, 2016.