Description Logics

Carlos Areces
carlos.areces@gmail.com

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What we do Today

- Structuring Knowledge

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- But

  Claim 4: *Classifying information is often a difficult and expensive task.*

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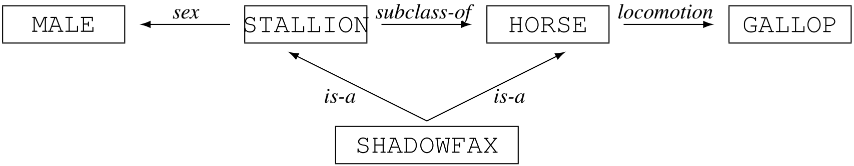
  - *Computational economy* refers to the efficiency of reasoning with such structures.
- The idea of developing knowledge representation systems based on a structured representation of knowledge has been pursued for a long time in Artificial Intelligence.

Semantic Networks



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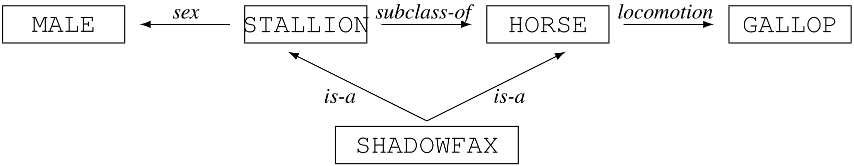

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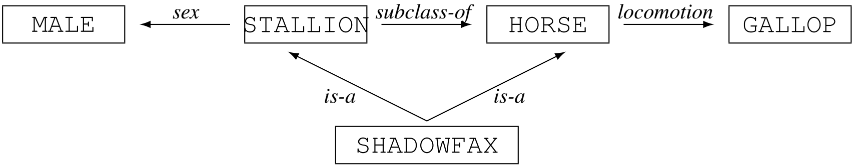

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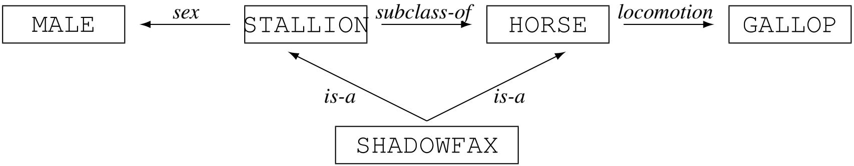

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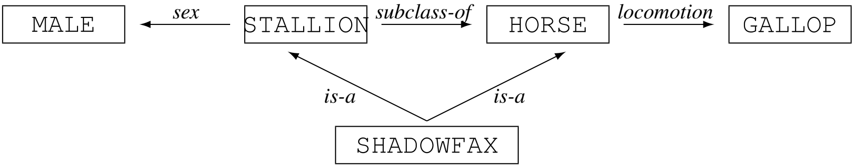

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- **And DLs were born:** Description logics (DLs) are a family of formal languages with a clearly specified semantics, usually in terms of first-order models, together with specialized inference mechanisms to account for knowledge classification.
  One of the original aims of the research in this field is to identify fragments of FO able to capture the features needed for representing a particular problem, and which can still allow for the design of efficient reasoning algorithms.

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  - With highly optimized inference systems.

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Example: The "Happy Father"

$$\mathsf{Concepts} = \{\ \mathsf{Man},\ \mathsf{Woman},\ \mathsf{Happy},\ \mathsf{Rich}\ \}$$

$$\mathsf{Roles} = \{\ \mathsf{has\text{-}children}\ \}$$

$$\mathsf{Individuals} = \{\ \mathsf{carlos}\ \}$$

$$\begin{aligned}\mathsf{HappyFather} \equiv \mathsf{Man}\ &\wedge \exists\, \mathsf{has\text{-}children.Man} \wedge \\ &\exists\, \mathsf{has\text{-}children.Woman} \wedge \\ &\forall\, \mathsf{has\text{-}children.(Happy} \vee \mathsf{Rich)}\end{aligned}$$

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Counterexample!!!

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  - To add 'semantic markup' to the information in the web.
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- Computational Linguistics
  - Many tasks in computational linguistics require inference and ‘background knowledge’: reference resolution, question/answering.
  - In some cases, the expressive power of DLs is enough and we don’t need to move to FOL.

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- Galen project replaced static hierarchy with DL
  - Describe concepts (e.g., spiral fracture of left femur)
  - Use DL classifier to build taxonomy
- Needed expressive DL and efficient reasoning
  - Descriptions used transitive roles, inverses, GCIs, etc.
  - Even prototype KB was very large ($\sim$ 3.000 concepts)
  - Existing classifier took $\sim$24 hours to classify KB
  - FaCT system takes $\sim$60 seconds.

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- DL reasoning can be used to support, e.g.,
  - Schema design and integration
  - Query optimization
  - Interoperability and federation

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- 4th Stage:
  - Mature implementations (Commercial!)
  - Applications and tools start to be widely used (e.g., Semantic Web).

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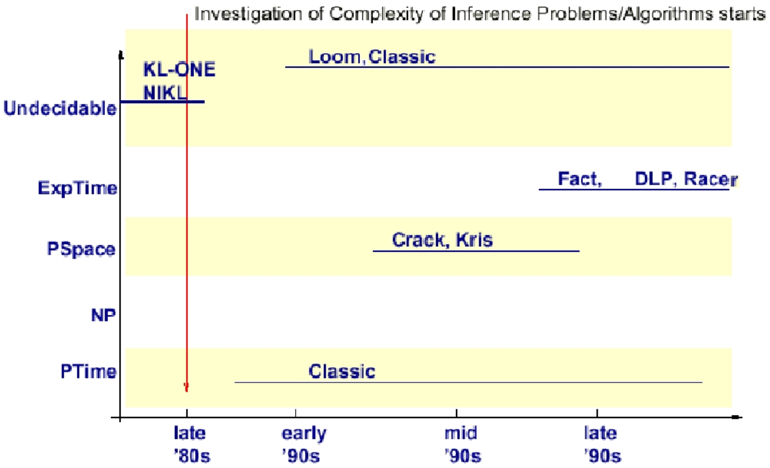

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- *Assertional information*: which records "specific" or "local" information, being true of certain particular individuals in the situation.
- All known information is then modeled as a pair $\langle T, A \rangle$, where T is a set of formulas concerning terminological information (the T-Box) and A is a set of formulas concerning assertional information (the A-Box).

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  - Assertions: assign concepts and roles to particular elements in our model: E.g., $\mathsf{carlos{:}\neg HappyFather}$
- A knowledge base is a pair $\langle D, A\rangle$ with D a set of definitions (the "T-Box") and A a set of assertions (la "A-Box").

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- A DL inference system takes a knowledge base $\langle D, A\rangle$ and can solve questions like:

$\langle D, A\rangle \models a : C$	"Given $\langle D, A\rangle$, is concept C applicable to a?"
$\langle D, A\rangle \models C \equiv D$	"Given $\langle D, A\rangle$, are concepts C and D quivalent?"

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$\exists R.C$	some element acc. through R is in C	$\exists$has-children.Woman

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$$\mathsf{GrandMother} \equiv \mathsf{Woman} \wedge \exists\mathsf{has\text{-}children}.\exists\mathsf{has\text{-}children}.\top$$

  (Equivalent to say both $C \sqsubseteq D$ and $D \sqsubseteq C$)

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- Assigning elements to relations: $(a, b){:}R$. Indicates that the elements a and b are related via the role R.

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- Assigning elements to relations: $(a, b){:}R$. Indicates that the elements a and b are related via the role R.

$$\text{(carlos,stanford):Lives-in}$$

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(i.e., a DL model is nothing else than an FO model for the signature $\langle \mathsf{CON} \cup \mathsf{ROL}, \{\}, \mathsf{IND} \rangle$)

Semantics: Concepts

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Given an interpretation $\mathcal{I}$ we can define the interpretation of an arbitrary $\mathcal{ALC}$ concept recursively as

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\begin{array}{rcl}
(\neg C)^{\mathcal{I}} & := & \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}} \\
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$C \sqcup D$ is equivalent to $\neg(\neg C \sqcap \neg D)$ and
$(\forall R.C)$ is equivalent to $\neg(\exists R.\neg C)$.

Semantics: Definitions and Assertions

Given an interpretation $\mathcal{I}$ we say that

- $\mathcal{I}$ satisfies a partial definition $C \sqsubseteq D$ (total definition $C \equiv D$) iff
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- $\mathcal{I}$ satisfies a KB $K = \langle T, A \rangle$ iff $\mathcal{I}$ satisfies T and A.

A KB K is consistent (or satisfiable) iff there is an interpretation $\mathcal{I}$ satisfying it.

A Complete Example

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$$
\begin{array}{rcl}
\mathsf{Woman} & \sqsubseteq & \mathsf{Person} \wedge \exists \mathsf{sex.Female} \\
\mathsf{Man} & \sqsubseteq & \mathsf{Person} \wedge \exists \mathsf{sex.Male} \\
\mathsf{FatherOrMother} & \equiv & \mathsf{Person} \wedge \exists \mathsf{has\text{-}children.Person} \\
\mathsf{Mother} & \equiv & \mathsf{Woman} \wedge \mathsf{FatherOrMother} \\
\mathsf{Father} & \equiv & \mathsf{Man} \wedge \mathsf{FatherOrMother}
\end{array}
$$

$$\mathsf{alice}:\mathsf{Mother}$$

$$(\mathsf{alice},\mathsf{betty}):\mathsf{has\text{-}children}$$

$$(\mathsf{alice},\mathsf{carlos}):\mathsf{has\text{-}children}$$

Syntax and Semantics: Summary

Constructor	Syntax	Semantics

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concept name	C	$C^{\mathcal{I}}$

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univ. quant.	$\forall R.C$	$\{d_1 \mid \forall d_2 \in \Delta^{\mathcal{I}}.(R^{\mathcal{I}} d_1 d_2 \rightarrow d_2 \in C^{\mathcal{I}})\}$

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number restr. ($\mathcal{N}$)	$\geq nR$	$\{d_1 \mid \lvert\{d_2 \mid R^{\mathcal{I}} d_1 d_2\}\rvert \geq n\}$
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one-of ($\mathcal{O}$)	$\{a_1, \ldots, a_n\}$	$\{d \mid d = a_i^{\mathcal{I}}$ for some $a_i\}$
filler ($\mathcal{B}$)	$\exists R.\{a\}$	$\{d \mid d = R^{\mathcal{I}} d a^{\mathcal{I}}\}$

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one-of ($\mathcal{O}$)	$\{a_1, \ldots, a_n\}$	$\{d \mid d = a_i^{\mathcal{I}}$ for some $a_i\}$
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Shadowfax, the Stallion

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a fact which is not explicit in the knowledge base.

Reasoning Tasks

Let Σ be a knowledge base, $C_1, C_2 \in \mathsf{CON}(\mathcal{L})$, $R \in \mathsf{ROL}(\mathcal{L})$ and $a, b \in \mathsf{IND}$, we define the following *reasoning tasks*

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Reasoning Tasks

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  - Many rules uses non-deterministic expansions $\Rightarrow$ search

Constraint Systems

- Let VAR be disjoint from IND, a *constraint* is a formula

$$s{:}C \mid (s,t){:}R \mid s \dot{\neq} t \mid \forall x.x{:}C,$$

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- Let $\mathcal{I}$ be an interpretation, an *$\mathcal{I}$-assignment* is a function α that maps every variable in VAR to an element of $\Delta^{\mathcal{I}}$ and every individual a in IND to $a^{\mathcal{I}}$.

Satisfaction of Constraints

- We use pairs $\langle \mathcal{I}, \alpha \rangle$ to define *satisfaction of constraints*. Let $s^{\langle \mathcal{I}, \alpha \rangle}$ be $s^{\mathcal{I}}$ if $s \in \mathsf{IND}$ and $\alpha(s)$ if $s \in \mathsf{VAR}$,

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- A *constraint system* CS is a finite, non-empty set of constraints. A pair $\langle \mathcal{I}, \alpha \rangle$ *satisfies* CS if $\langle \mathcal{I}, \alpha \rangle$ satisfies every constraint in CS, in which case we say that CS is *satisfiable*.

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- And given a knowledge base Σ, and an assertion $a{:}C$

$$\Sigma \models a{:}C \text{ iff } \mathsf{CS}_\Sigma \cup \{a{:}\neg C\} \text{ is unsatisfiable.}$$

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Corollary: The tableau algorithm is a PSPACE decision procedure for consistency (and subsumption) of $\mathcal{ALCN}$ concepts. And $\mathcal{ALCN}$ has the tree property.

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Proof of Lemma:

1. (Termination) The algorithm "increasingly" constructs a tree whose

   depth is linear in $|C|$: quantifier depth decreases from node to succs.

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3 Obvious: T with a clash has no model – recall definition of a clash

$$\begin{array}{rcl} \{A, \neg A\} & \subseteq & \mathcal{L}(x)\text{or} \\ \{(\geq mR), (\leq nR)\} & \subseteq & (x) \text{ for } n < m \end{array}$$

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- $\rightsquigarrow$ canonical tree model for input concept.

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- construct/search the tree **depth first**

PSPACE

- To make the tableau algorithm run in PSPACE, start by recalling Savitch: PSPACE = NPSPACE
- Observe that branches are independent from each other
- Observe that each node (label) requires linear space only
- recall that path are of lenght $\leq |C|$
  $\rightsquigarrow$ each path can be stored in $\mathcal{O}(|C^2|)$
- construct/search the tree **depth first**
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lazy unfolding	(unfolding on demand) is optimal, consistency is PSPACE decidable

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Use **blocking** (cycle detection) to ensure termination
(but the right blocking to not destroy soundness and completeness)

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  - Computing *subsumption* between concepts
    - Objective is to minimize cost of single subsumption tests
    - Small number of hard tests can dominate classification time
    - Latest DL research has mainly addressed this problem.

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  - But often generally applicable to search based algorithms

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- Heuristic ordering of propositional and modal expansions
  - Minimize/maximize constrainedness (e.g., MOMS)
  - Maximize backtracking (e.g., oldest first)

Normalization and Simplification

- Normalize concepts to standard form, e.g.:
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  - $(D \sqcap C) \sqcap (A \sqcap D) \longrightarrow A \sqcap C \sqcap D$
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- Lazily unfold concepts in tableaux algorithm
  - Use name/pointers to refer to complex concepts
  - Only add structure as required by the progress of the algorithm
  - Detect clashes between lexically equivalent concepts

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- Reasoning w.r.t. primitive definition axioms is relatively efficient
  - For $CN \sqsubseteq D$, add D **only** to node labels containing CN
  - for $CN \sqsupseteq D$, add $\neg D$ **only** to node labels containing $\neg CN$
  - We can expand definitions lazily
    - Only add definitions **after** other local (propositional) expansions.
    - Only add definitions one step at a time.

Absorption II

- Transform GCIs into primitive definitions, e.g.:
  - $CN \sqcap C \sqsubset D \longrightarrow CN \subseteq D \sqcup \neg C$
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- Absorb into existing primitive definitions, e.g.:
  - $CN \sqsubseteq A, CN \sqsubseteq D \sqcup \neg C \longrightarrow CN \sqsubseteq A \sqcap (D \sqcup \neg C)$
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- Use lazy expansion techniques with primitive definitions
  - Disjunctions only added to “relevant” node labels
- Performance improvements often exceptional
  - At least **four orders of magnitude** with GALEN

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  - Performance improvements again very good.

Backjumping

E.g if $\exists R.\neg A \sqcap \forall R(A \sqcap B) \sqcap (C_1 \sqcup D_1) \sqcap \ldots \sqcap (C_n \sqcap D_n) \in \mathcal{L}(x)$

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  - Care required when used with blocking or inverse roles
  - Significant performance gains with some kinds of problem
- Cache (partial) models of concepts
  - Use to detect "obvious" non-subsumption
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  - Identical node labels often recur during expansion
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  - If not, continue with standard subsumption test.