# LogicS

## Lecture #3: How many Angels can Dance on the Head of a Pin?

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# The Story so Far

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  2) How much does it cost?

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- In particular, we will show that we can code the Graph Coloring problem.

# What do we do Today?

How Many Angels can Dance on the Head of a Pin?

Or, how much can you encode in a 1-point structure?

- We will see that we can code quite complex problems in PL.
- In particular, we will show that we can code the Graph Coloring problem.
- Then, we will introduce an efficient algorithm for deciding satisfiability of PL-SAT: the Davis Putnam algorithm.

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  - planning (e.g., graphplan).
- Note that these problems have real world applications!

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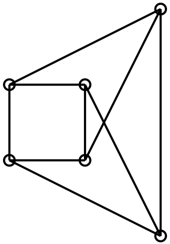

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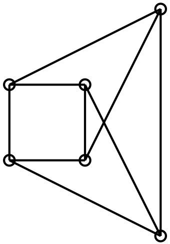

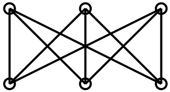

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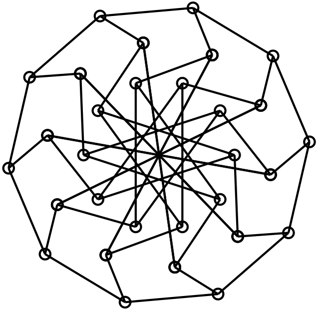

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3. Neighboring nodes have different colors. $\neg p_{il} \vee \neg p_{jl}$,
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- Using the encoding in the previos page we efectively obtain for each graph $G$ and $k$ color, a formula $\varphi_{G,k}$ in $PL$ such that

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  - Always correctly.
- The best known complete methods probably are
  - truth tables
  - tableaux
  - axiomatics, Gentzen calculi, natural deduction, resolution
  - Davis-Putnam

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The Diplomatic Problem:

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if Σ={} then return SAT                // (SAT)
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Choose literal l and
   if DP(Σ[{l=true}]) return SAT
      then return SAT
      else return DP(Σ[{l=false}])             // (Split)
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$$\{\{P, \neg Q\}, \{Q, R\}, \{\neg R \vee \neg P\}\}$$

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- Sadly real world applications easily go into the thousands of variables (remember coloring: #nodes $\times$ #colors).
- But this is worst time complexity. You might get lucky...

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- Examples of these methods are GSAT and WalkSAT.
- For example, a $k$-coloring algorithm based on GSAT was able to beat specialized coloring algorithms.

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    (able to cope with problems with hundreds of propositional symbols, but our codings easily get into the thousands).
- Still, no matter how nicely we paint them, 1-point relational structures are booooooooooring.

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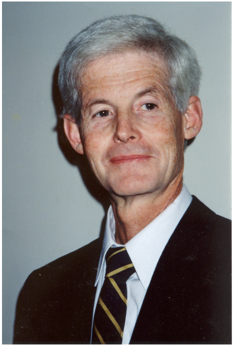

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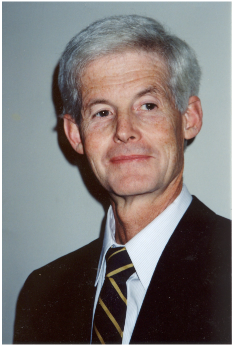

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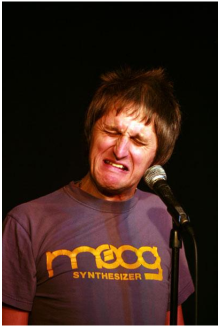

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Lecture #4

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