LogicS

Lecture #7: DIY First-Order Logic

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INRIA Nancy Grand Est
Nancy, France

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The Story so Far

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- What do you think? Can we mix the First Order Recipe?

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- We will see what we can reuse of what we already have. . .
- . . . and extend the language if necessary.
- We will then show that the language we obtain is actually equivalent to the 'classical' First Order Language.

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  - Constants
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  - The counting operators $\langle = n\ R \rangle$
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- First order quantification gives as a delicate control (via variables) of what we are quantifying on.

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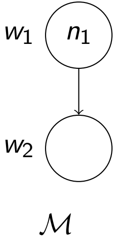

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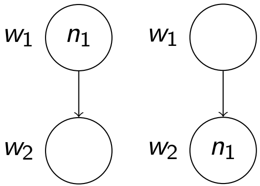

$\mathcal{M} \Rightarrow \mathcal{M}\{n_1 \mapsto w_2\}$

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We can think of n as an FO variable now.

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- If you really understand how one works, you already know how the other does.

Relevant Bibliography

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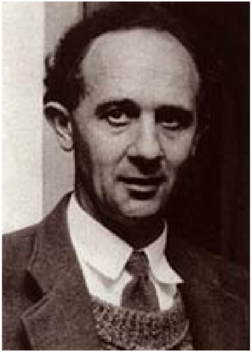

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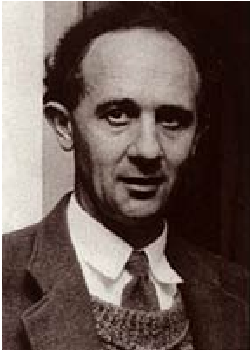

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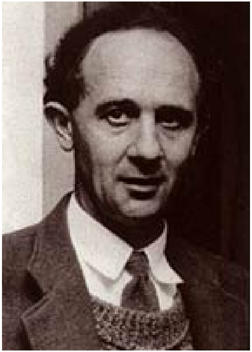

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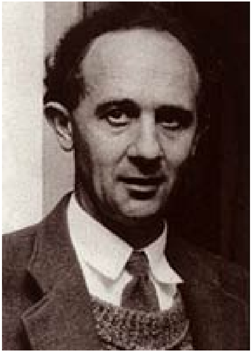

Prior, Arthur (1967). *Chapter V.6 of Past, Present and Future*. Clarendon Press, Oxford.

The Next Lecture

We Like it Complete and Compact

(and We have a Soft Spot for Löwenheim-Skolem)