# LogicS

## Lecture #8: Putting Tiles in an Infinite Bathroom

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# The Story so Far

- We put together various bits-and pieces we have played with (names, :, diamonds) added the $\langle\langle x\rangle\rangle$ and $[\![x]\!]$ operators, and reached the most expressive language we have seen so far.
- We reached "first-order logic", the logic often considered to be "classical logic", and one of the most distinctive and important spots on the logical landscape. For our $\langle\langle x\rangle\rangle$ and $[\![x]\!]$ are really just the familiar $\exists$ and $\forall$ quantifiers.
- We've come a long way. Think that on Monday we were in propositional logic!

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- We will argue that the problem is undecidable
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- Actually, we are going to show how to tile an infinite bathroom.

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The halting problem of Turing machines is the standard example of an undecidable problem. The behaviour of a Turing machine, and the predicate that says that a given turing machine stops on all inputs can be expressed in FO.

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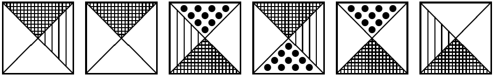

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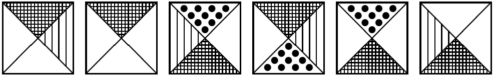

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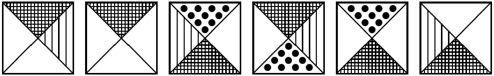

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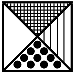

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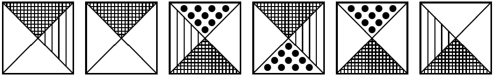

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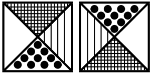

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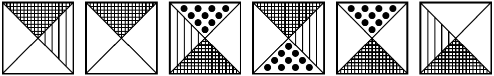

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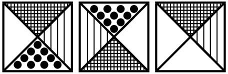

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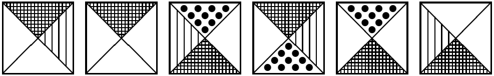

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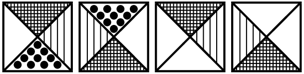

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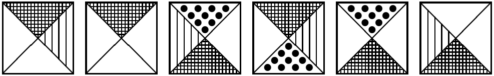

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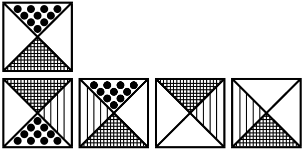

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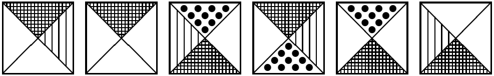

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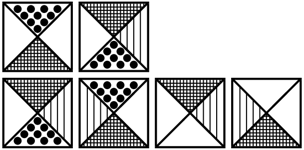

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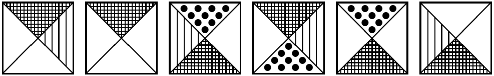

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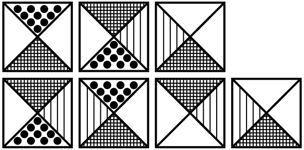

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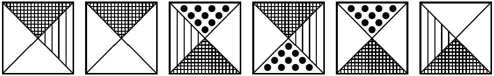

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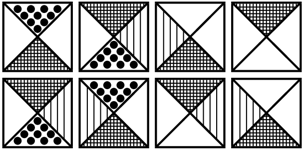

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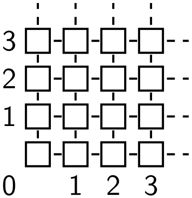
- Covering $\mathbb{N} \times \mathbb{N}$
  - tiling $\mathbb{N} \times \mathbb{N}$: Given a finite set of tiles $\mathcal{T}$, can $\mathcal{T}$ cover $\mathbb{N} \times \mathbb{N}$?
  - this problem is undecidable (It is equivalent to the halting problem of Turing machines)



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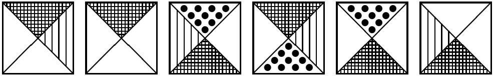

$t_1$ $t_2$ $t_3$ $t_4$ $t_5$ $t_6$

we will have

$$\begin{array}{lcl} H & = & \{(t_1, t_3), (t_1, t_6), (t_2, t_4), (t_2, t_5), (t_2, t_1), \ldots\} \\ V & = & \{(t_1, t_3), (t_1, t_5), (t_1, t_6), (t_2, t_3), (t_2, t_5), \ldots\} \end{array}$$

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  Then $\varphi$ is satisfiable iff $\mathcal{T}$ covers $\mathbb{N} \times \mathbb{N}$.

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- He showed that the halting problem could not be decided by a Turing Machine.
- And that the behavior of a Turing Machine can easily be described in first-order logic, providing an alternative proof that the satisfiability problem of first-order logic is undecidable.

Turing, Alan (1936), *On Computable Numbers, with an Application to the Entscheidungsproblem*. Proceedings of the London Mathematical Society, Series 2, Vol.42, pp 230–265, http://www.thocp.net/biographies/papers/turing_oncomputablenumbers_1936.pdf